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Indices for superconformal field theories in 3, 5 and 6 dimensions

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Abstract: We present a trace formula for a Witten type Index for superconformal field theories in $d = 3, 5$ and 6 dimensions, generalizing a similar recent construction in $d = 4$. We perform a detailed study of the decomposition of long representations into sums of short representations at the unitarity bound to demonstrate that our trace formula yields the most general index (i.e. quantity that is guaranteed to be protected by superconformal symmetry alone) for the corresponding superalgebras. Using the dual gravitational description, we compute our index for the theory on the world volume of N M2 and M5 branes in the large N limit. We also compute our index for recently constructed Chern Simons theories in three dimensions in the large N limit, and find that, in certain cases, this index undergoes a large N phase transition as a function of chemical potentials.

Keywords: AdS-CFT Correspondence, Extended Supersymmetry, Supersymmetric gauge theory.

Contents

1. Introduction

Supersymmetric fixed points of the renormalization group equations are believed to be always either free or superconformally invariant. Thus the IR/UV behaviour of any supersymmetric field theory, if nontrivial, is governed by a superconformal fixed point. Consequently, the study of superconformal dynamics has a special place in the study of supersymmetric field theories.

In radial quantization, the Hilbert space of any unitary superconformal field theory may be decomposed into a direct sum over irreducible unitary, lowest energy representation of the superconformal algebra. Such representations have been classified in every dimension (see [1 − 5] and references therein); the list of these representations turn out to include a special set of BPS representations. These representations are called 'short' because they have fewer states than generic representations (we explain this more precisely below); they also have the property that the energies of all states they host are determined by the other conserved charges that label the representation.

Consider any fixed line of superconformal field theories labeled by some continuous 'coupling constant' λ. Suppose that, at any given value of λ, the Hilbert space of the theory possesses some states that transform in short representations of the superconformal algebra. Under an infinitesimal variation of λ the energies of the corresponding states can only change if some of these representations jump from being short to long. However short representation always contain fewer states than long representations with (almost) the same quantum numbers. As a consequence, the jump of a single BPS representation from short to long is inconsistent with the continuity of the spectrum of the theory as a function of λ. Indeed such jumps are consistent with continuity only when they occur simultaneously for a group of short representations that have the property that their state content is identical to the content of a long representation. Such a bunch of BPS representations can continuously be transmuted into a long representation, after which the energies of its constituent states can be renormalized.

Consequently, a detailed study of all possible ways in which short representations can combine up into long representations permits the the classification of superconformal indices for superconformal field theories.[1] In this paper we perform this study for superconformal algebras in $d = 3, 5$ and 6 and use our results to provide a complete classification of all superconformal indices in these dimensions. In each of these cases, we also provide a trace formula that, when evaluated in a superconformal field theory, may be used to extract all these superconformal indices. This is the analogue of the trace formula described in [6]

[1] By a superconformal index we mean any function of the spectrum that is forced by the superconformal algebra to remain constant under continuous variations of the spectrum.

for the Witten index. Thus the Witten index we define in this paper constitutes the most general superconformal Index in $d = 3, 5, 6$.[2]

We then proceed to compute our superconformal Witten Index for specific superconformal field theories. We first perform this computation for the superconformal field theories on the world volume of N M2 and N M5 branes, at $N = 1$ (using field theory) and at $N = \infty$ (using the dual supergravity description). We find that our index has significant cancellations compared to the simple partition function over supersymmetric states. In each case, the density of states in the Index grows slower in comparison to the supersymmetric entropy. We also compute our index for some of the Chern Simons superconformal field theories recently analyzed by Gaiotto and Yin [10]; and find that, in some cases, this index undergoes a large N phase transition as a function of chemical potentials.

This paper is divided into 3 self-contained parts. Superconformal algebras in $d = 3$ are analyzed in section 2, in $d = 6$ are discussed in section 3 and in $d = 5$ are discussed in 4. In each section, we describe the relevant algebra and its unitary representations. We then discuss short representations and enumerate all possible ways in which short representations can pair up into long representations. We use this enumeration to provide, in each dimension, an exhaustive list of all indices that are protected by group theory alone. We also provide a trace formula for a Witten type index that may be evaluated via a path integral. These indices count states that are annihilated by a particular supercharge. We discuss how the Witten Index may be expanded out in characters of the subalgebra of the superconformal algebra that commutes with this supercharge. The coefficients of these characters in the Witten Index are nothing but the indices mentioned above.

In $d = 3$, we evaluate our index in three different theories: (a) Supergravity on $AdS_4 \times S^7$ (b) the worldvolume theory of a single $M2$ brane and (c) the Chern-Simons matter theories recently discussed in [10]. In $d = 6$, we evaluate our index in supergravity on $AdS_7 \times S^4$ and in the worldvolume theory of a single $M5$ brane.

Finally, we wish to mention a subtlety that we have avoided in our discussion above. Indices may fail to be protected if the spectrum of the theory contains a continuum [6, 11] or is singular for some parameters. Lately, this issue has attracted interest in the context of 2 dimensional conformal field theories and we direct the reader to [12−14] for some recent discussions.

2. d=3

2.1 The superconformal algebra and its unitary representations

The bosonic subalgebra of the $d = 3$ superconformal algebra is $\mathrm{SO}(3,2)\times\mathrm{SO}(n)$ (the conformal algebra times the R symmetry algebra). The anticommuting generators in this algebra may be divided into the generators of supersymmetry (Q) and the generators of superconformal symmetries (S). Supersymmetry generators transform in the vector representation

[2] The corresponding results are already known in $d = 4$ [7]. In 2 dimensions the analogue of the indices we will study here is the famous 'elliptic genus' [8, 9] while superconformal algebras do not exist in $d > 6$.

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of the R-symmetry group SO(n),[3] have charge half under dilatations (the SO(2) factor of the compact SO(3) × SO(2) ∈ SO(3, 2)) and are spinors under the SO(3) factor of the same decomposition. Superconformal generators $S_i^\mu = (Q^i_\mu)^\dagger$ transform in the spinor representation of SO(3), have scaling dimension (dilatation charge) $(-\frac{1}{2})$, and also transform in the vector representation of the R-symmetry group. In our notation for supersymmetry generators i is an SO(3) spinor index while μ is an R symmetry vector index.

We pause to remind the reader of the structure of the commutation relations and irreducible unitary representations of the $d = 3$ superconformal algebra (see [4] and references therein). As usual, the anticommutator between two supersymmetries is proportional to momentum times an R symmetry delta function, and the anticommutator between two superconformal generators is obtained by taking the Hermitian conjugate of these relations. The most interesting relationship in the algebra is the anticommutator between Q and S. Schematically

$$\{S_i^\mu, Q^j_\nu\} \sim \delta^\mu_\nu T_i^j - \delta_i^j M^\mu_\nu$$

Here T_i^j are the U(2) ∼ SO(3) × SO(2) generators and M^μ_ν are the SO(n) generators.

Irreducible unitary lowest energy representations of this algebra possess a distinguished set of lowest energy states called primary states. Primary states have the lowest value of ϵ_0 — the eigenvalue of the dilatation (or energy) operator — of all states in their representation. They transform in irreducible representation of SO(3) × SO(n), and are annihilated by all superconformal generators and special conformal generators.[4]

Primary states are special because all other states in the unitary (always infinite dimensional) representation may be obtained by acting on the primary with the generators of supersymmetry and momentum. For a primary with energy ϵ_0, a state obtained by the action of k different Q s on the primary has energy $\epsilon_0 + \frac{k}{2}$, and is said to be a state at the k^{th} level in the representation. The energy, SO(3) highest weight (denoted by $j = 0, \frac{1}{2}, 1\ldots$) and the R-symmetry highest weights $(h_1, h_2 \ldots h_{[n/2]})$ [5] of primary states form a complete set of labels for the entire representation in question.

Any irreducible representation of the superconformal algebra may be decomposed into a finite number of distinct irreducible representations of the conformal algebra. The latter are labeled by their own primary states, which have a definite lowest energy and transform in a given irreducible representation of SO(3). The state content of an irreducible representation of the superconformal algebra is completely specified by the quantum numbers of its constituent conformal primaries.

As we have mentioned in the introduction, the superconformal algebra possesses special short or BPS representations which we will now explore in more detail. Consider a representation of the algebra, whose primary transforms in the spin j representation of SO(3) and in the SO(n) representation labeled by highest weights $\{h_i\}$ $i = 1, \cdots, \left[\frac{n}{2}\right]$.

[3] In the literature on the worldvolume theory of the $M2$ brane, the supercharges are taken to transform in a spinor of SO(8). This is consistent with the statement above, because for $n = 8$, the vector and spinor representations are related by a triality flip and a change of basis takes one to the other.

[4] i.e. all generators of negative scaling dimension.

[5] h_i are eigenvalues under rotations in orthogonal 2 planes in R^n. Thus, for instance, $\{h_i\} = (1, 0, 0, \ldots 0)$ in the vector representation.

We normalize primary states to have unit norm. The superconformal algebra — plus the Hermiticity relation $(Q^i_\mu)^\dagger = S^\mu_i$ — completely determines the inner products between any two states in the representation. All states in an unitary representation must have positive norm: however this requirement is not algebraically automatic, and, in fact imposes a nontrivial restriction on the quantum numbers of primary states. This restriction takes the form $\epsilon_0 \geq f(j, h_i)$ as we will now explain.[6]

Let us first consider descendant states, at level one, of a representation whose primary has SO(3) and SO(n) quantum numbers $j, (h_1 \dots h_{[n/2]})$. It is easy to compute the norm of all such states by using the commutation relations of the algebra. As long as $j \neq 0$ it turns out that the level one states with lowest norm transform in in the spin $j - \frac{1}{2}$ representation of the conformal group and in the $(h_1+1, \{h_i\})$ $i = 2, \cdots, \left[\frac{n}{2}\right]$ representation of SO(n) [4]. The highest weight state in this representation may be written explicitly as (see [16])

$$|Zn_1\rangle = A_1^- |h.w\rangle \equiv \left(Q_1^{-\frac{1}{2}} - Q_1^{\frac{1}{2}} J_- \left(\frac{1}{2J_z}\right)\right) |h.w\rangle \tag{2.1}$$

where J_- denotes the spin lowering operator of SO(3) and $Q_1^{\pm\frac{1}{2}}$ are supersymmetry operators with $j = \pm\frac{1}{2}$ and $(h_1, h_2, \dots h_{[n/2]}) = (1, 0, \dots, 0)$. Here $|h.w\rangle$ is a highest weight state with energy ϵ_0, SU(2) charge j and SO(n) charge $(h_1, h_2, \dots, h_{[n/2]})$. The norm of this state is easily computed and is given by,

$$\langle Zn_1 | Zn_1\rangle = \left(1 + \frac{1}{2j}\right)(\epsilon_0 - j - h_1 - 1) \tag{2.2}$$

It follows that the non negativity of norms of states at level one (and so the unitarity of the representation) requires that the charges of the primary should satisfy

$$\epsilon_0 \geq j + h_1 + 1 \tag{2.3}$$

For $j \neq 0$ this inequality turns out to be the necessary and sufficient condition for a representation to be unitary.

When the primary saturates the bound (2.3) the representation possess zero norm states: however it turns out to be consistent to define a truncated representation by simply deleting all zero norm states. This procedure yields a physically acceptable representation whose quantum numbers saturate (2.3). This truncated representation is unitary (has only positive norms) but has fewer states than the generic representation, and so is said to be 'short' or BPS.

The set of zero norm states we had to delete, in order to obtain the BPS representation described above, themselves transform in a representation of the superconformal algebra. This representation is labeled by the primary state $|Zn_1\rangle$ (see (2.1)).

Let us now turn to the special case $j = 0$. In this case $|Zn_1\rangle$ is ill defined and does not exist; no states with its quantum numbers occur at level one. The states of lowest norm at level one transform in the spin half SO(3) representation, and have SO(n) highest

[6] These techniques have been used in the investigation of unitarity bounds for conformal and superconformal algebras in [15, 1–4, 16].

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weights $h'_1 = h_1 + 1, \{h_i\}\ i = 2, \cdots, \frac{n}{2}$. The highest weight state in this representation is $|Zn_2\rangle = A_1^+|h.w.\rangle \equiv Q_1^{\frac{1}{2}}|h.w\rangle$. The norm of this state is $(\epsilon_0 - h_1)$. Unitarity thus imposes the constraint $\epsilon_0 \geq h_1$. However, in this case, this condition is necessary but not sufficient to ensure unitarity, as we now explain.

As we have remarked above, the state $|Zn_1\rangle = A_1^-|h.w\rangle$ is ill defined when $j = 0$. However $|s_2\rangle = \left(A_1^+ A_1^-\right)|h.w\rangle = Q_1^{\frac{1}{2}} Q_1^{-\frac{1}{2}}|h.w\rangle$ is well defined even in this situation (when $j = 0$). The norm of this state is easily computed and is given by,[7]

$$\langle s_2|s_2\rangle = (\epsilon_0 + j - h_1)(\epsilon_0 - j - h_1 - 1). \tag{2.4}$$

It follows that, at $j = 0$, the positivity of norm of all states requires either that $\epsilon_0 \geq h_1 + 1$ or that $\epsilon_0 = h_1$. This turns out to be the complete set of necessary and sufficient conditions for the existence of unitary representations. Representations with $j = 0$ and $\epsilon_0 = h_1 + 1$ or $\epsilon_0 = h_1$ are both short. The representation at $\epsilon_0 = h_1$ is an isolated short representation since there is no representation in the energy gap $h_1 \leq \epsilon_0 \leq (h_1 + 1)$; its first zero norm state occurs at level one. The first zero norm state in the $j = 0$ representation at $\epsilon_0 = h_1 + 1$ occurs at level 2 and is given by $|s_2\rangle$.

In summary, short representations occur when the highest weights of the primary state satisfy one of the following conditions [4].

$$\begin{aligned} \epsilon_0 =& j + h_1 + 1 \quad \text{when j} \geq 0, \\ \epsilon_0 =& h_1 \quad \text{when j} = 0. \end{aligned} \tag{2.5}$$

The last condition gives an isolated short representation.

2.2 Null vectors and character decomposition of a long representation at the unitarity threshold

As we have explained in the previous subsection, short representations of the $d = 3$ superconformal algebra are of two sorts. The energy of a 'regular' short representation is given by $\epsilon_0 = j + h_1 + 1$. The null states of this representation transform in an irreducible representation of the algebra. When $j \neq 0$ the highest weights of the primary at the head of this null irreducible representation is given in terms of the highest weights of the representation itself by $\epsilon'_0 = \epsilon_0 + \frac{1}{2}, \quad j' = j - \frac{1}{2}, \quad h'_1 = h_1 + 1, \quad h'_i = h_i$. Note that $\epsilon'_0 - j' - h'_1 - 1 = \epsilon_0 - j - h_1 - 1 = 0$, so that the null states also transform in a regular short representation. As the union of the ordinary and null states of such a short representation is identical to the state content of a long representation at the edge of the unitarity bound, we conclude that

$$\lim_{\delta \to 0} \chi[j + h_1 + 1 + \delta, j, h_1, h_j] = \chi[j + h_1 + 1, j, h_1, h_j] + \chi\left[j + h_1 + 3/2, j - \frac{1}{2}, h_1 + 1, h_j\right] \tag{2.6}$$

[7] When $j \neq 0$, the norm of $|s_2\rangle$ had to be proportional to $(\epsilon_0 - j - h_1 - 1)$ simply because the norm of $|s_2\rangle$ must vanish whenever $|Zn_1\rangle$ is of zero norm. The algebra that leads to this result is correct even at $j = 0$ (i.e. when $|Zn_1\rangle$ is ill defined).

where $\chi[\epsilon_0, j, h_i]$ denotes the supercharacter of the irreducible representation with energy ϵ_0, SO(3) highest weight j and SO(n) highest weights $\{h_i\}$. Note that the χ s appearing on the r.h.s. of (2.6) are the supercharacters corresponding to short representations.

On the other hand when $j = 0$ the null states of the regular short representation occur at level 2 and are labelled by a primary with highest weights $\epsilon_0' = \epsilon_0 + 1, \quad j' = 0, h_1' = h_1 + 2, \quad h_i' = h_i$. Note in particular that $j' = 0$ and $\epsilon_0' - h_1' = \epsilon_0 - h_1 - 1 = 0$. It follows that the null states of this representation transform in an isolated short representation, and we conclude

$$\lim_{\delta \to 0} \chi[h_1 + 1 + \delta, j = 0, h_1, h_j] = \chi[h_1 + 1, j = 0, h_1, h_j] + \chi[h_1 + 2, j = 0, h_1 + 2, h_j] \quad (2.7)$$

Recall that isolated short representations are separated from all other representations with the same SO(3) and SO(n) quantum numbers by a gap in energy. As a consequence it is not possible to 'approach' such representations with long representations; consequently we have no equivalent of (2.7) at energies equal to $h_1 + \delta$.

For use below we define some notation. We will use $c(j, h_i)$ (with $i = 1, 2, \ldots, [\frac{n}{2}]$) to denote a regular short representation with SO(3) and SO(n) highest weights j, h_i respectively and $\epsilon_0 = j + h_1 + 1$ (when $j \geq 0$). We will also use the symbol $c(-\frac{1}{2}, h_1, h_j)$ (with $h_1 \geq h_2 - 1$) to denote the isolated short representation with SO(3) quantum number 0, SO(n) quantum numbers $h_1 + 1, h_j$ (here $j = 2, 3, \ldots, [\frac{n}{2}]$) respectively and $\epsilon_0 = h_1 + 1$. The utility of this notation will become apparent below.

2.3 Indices

The state content of any unitary superconformal quantum field theory may be decomposed into a sum of an (in general infinite number of) irreducible representations of the superconformal algebra. This state content is completely determined by specifying the number of times any given representation occurs in this decomposition. Consider any linear combination of the multiplicities of short representations. If this linear combination evaluates to zero on every collection of representations that appears on the r.h.s. of each of (2.6) and (2.7) (for all values of parameters), it is said to be an index. It follows from this definition that indices are unaffected by all possible pairing up of short representations into long representations, and so are invariant under any deformation of superconformal Hilbert space under which the spectrum evolves continuously. We now proceed to list these indices.

1. The simplest indices are simply given by the multiplicities of representations in the spectrum that never appear on the r.h.s. of (2.7) and (2.6) (for any values of the quantum numbers of the long representations on the l.h.s. of those equations). All such representations are easy to list; they are SO(3) singlet isolated representations whose SO(n) quantum number $h_1 - |h_2| \leq 1$ where h_1 and h_2 are both either integers or half integers, and $h_1 \geq |h_2| \geq 0$.

2. We can also construct indices from linear combinations of the multiplicities of representations that do appear on the r.h.s. of (2.7) and (2.6). The complete list of such

linear combinations is given by

$$I_{M,\{h_j\}} = \sum_{p=-1}^{M-|h_2|} (-1)^{p+1} n\left\{c\left(\frac{p}{2}, M-p, h_j\right)\right\}, \tag{2.8}$$

where $n[R]$ denotes the multiplicities of representations of type R and the Index label M is the value of h_1+2j for every regular short representation that appears in the sum above. Thus $M \geq |h_2|$ and both M and h_2 are either integers or half-integers.Also the set $\{h_j\}$ must satisfy the condition $h_2 \geq h_3 \ldots\ldots \geq |h_{[\frac{n}{2}]}|$ where all the h_i are either integers or all are half-integers.

2.4 Minimally BPS states: distinguished supercharge and commuting superalgebra

We will now describe states that are annihilated by at least one supercharge and its conjugate. Consider the special supercharge Q with charges $(j = -\frac{1}{2}, h_1 = 1, h_i = 0, \epsilon_0 = \frac{1}{2})$. Let $S = Q^\dagger$; it is easily verified that

$$\{S, Q\} = \Delta = \epsilon_0 - (h_1 + j) \tag{2.9}$$

Below we will be interested in a partition function over states annihilated by Q. Clearly all such states transform in irreducible representations of that subalgebra of the superconformal algebra that commutes with Q, S and hence Δ. This subalgebra is easily determined to be a real form of the supergroup $D(\frac{n-2}{2}, 1)$ or $B(\frac{n-3}{2}, 1)$, depending on whether n is even or odd. We follow the same notation as [4].

The bosonic subgroup of this commuting superalgebra is $\mathrm{SO}(2,1) \times \mathrm{SO}(n-2)$. The usual Cartan charge of $\mathrm{SO}(2,1)$ (the $\mathrm{SO}(2)$ rotation) and the Cartan charges of $\mathrm{SO}(n-2)$ are given in terms of the Cartan elements of the parent superconformal algebra by

$$E = \epsilon_0 + j, \quad H_i = h_{i+1} \quad \left(\text{with i} = 1, 2, \ldots, \left[\frac{\text{n}-2}{2}\right]\right). \tag{2.10}$$

2.5 A Trace formula for the general Index and its character decomposition

Let us define the Witten index

$$I^W = \mathrm{Tr}_R[(-1)^F \exp(-\beta\Delta + G)], \tag{2.11}$$

where the trace is evaluated over any Hilbert space R that hosts a representation (not necessarily irreducible) of the superconformal algebra. Here F is the Fermion number operator; by the spin statistics theorem $F = 2j$ in any quantum field theory. G is any element of the subalgebra that commutes with $\{S, Q, \Delta\}$; by a similarity transformation, G may be rotated into a linear combination of the Cartan generators of the subalgebra.

The Witten Index (2.11) receives contributions only from states that are annihilated by both Q and S (all other states yield contributions that cancel in pairs) and have $\Delta = 0$. So, it is independent of ζ. The usual arguments [6] also ensure that I^W is an index; consequently it must be possible to expand I^W as a linear sum over the indices defined in

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the previous section. In fact it is easy to check that for any representation A(reducible or irreducible),

$$I^W(A) = \sum_{M,\{h_i\}} I_{M,\{h_i\}} \chi_{\rm sub}(M+2,h_i) + \sum_{\{h_j\},h_1-|h_2|=0,1} n\left\{c\left(-\frac{1}{2},h_1-1,h_i\right)\right\}\chi_{\rm sub}(h_1,h_i). \tag{2.12}$$

where $\chi_{\rm sub}(E,H_i)$ (with $i=1,2,\ldots,[\frac{n-2}{2}]$) is the supercharacter of the subalgebra[8] with E and H_i being the highest weights of a representation of the subalgebra in the convention defined by (2.10). In the first term on the r.h.s. of (2.12) the sum runs over all the values of $M,\{h_j\}$ for which $I_{M,\{h_j\}}$ is defined (see below (2.8)). In the second term the sum runs over all the values of the set $\{h_j\}$ such that $h_2 \geq h_3 \ldots\ldots \geq |h_{[\frac{n}{2}]}|$. In order to obtain (2.13) we have used

$$I^W(c(j,h_1,h_j)) = (-1)^{2j+1}\chi_{\rm sub}(2j+h_1+2,h_i) \tag{2.13}$$

$$I^W\left(c\left(j=-\frac{1}{2},h_1,h_j\right)\right) = \chi_{\rm sub}(h_1+1,h_j) \tag{2.14}$$

Equation (2.13) asserts that the set of $\Delta = 0$ states (the only states that contribute to the Witten Index) in any short irreducible representation of the superconformal algebra transform in a single irreducible representation of the commuting subalgebra. In the case of regular short representations, the primary of the full representation has $\Delta = 1$. The primary of the subalgebra is obtained by acting on the primary of the full representation with a supercharge with quantum numbers ($j=\frac{1}{2}, h_1=1, h_i=0, \epsilon_0=\frac{1}{2}, \Delta=-1$). On the other hand the highest weight of an isolated superconformal short primary itself has $\Delta = 0$, and so is also a primary of the commuting sub super algebra. Equation (2.12) follows immediately from these facts.

Note that every index that appears in the list of subsection 2.3 appears as the coefficient of a distinct subalgebra supercharacter in (2.12). As supercharacters of distinct irreducible representations are linearly independent, it follows that knowledge of I^W is sufficient to reconstruct all superconformal indices of the algebra. In this sense (2.12) is the most general index that is possible to construct from the superconformal algebra alone.

2.6 The Index over M theory multi gravitons in $AdS_4 \times S^7$

We will now compute the Witten Index defined above in specific examples of three dimensional superconformal field theories. In this subsection we focus on the world volume theory of the M2 brane in the large N limit. The corresponding theory has supersymmetries and 16 superconformal symmetries. The bosonic subgroup of the relevant superconformal algebra is SO(3,2)×SO(8). We take the supercharges to transform in the vector representation of SO(8); this convention is related to the one used in much of literature on this theory by a triality flip.

[8]The supercharacter of a representation R is defined as $\chi_{\rm sub}(R) = {\rm tr}_R(-1)^F {\rm tr}\, e^{\mu\cdot\mathbf{H}}$, where $\mu\cdot\mathbf{H}$ is some linear combination of the Cartan generators specified by a chemical potential vector μ. F is defined to anticommute with Q and commute with the bosonic part of the algebra. The highest weight state is *always taken* to have $F=0$.

In the strict large N limit, the index over the M2 brane conformal field theory is simply the index over the Fock space of supergravitons for M theory on $AdS_4 \times S^7$ [17, 18]. In order to compute this quantity we first compute the index over single graviton states; the index over multi gravitons is given by the appropriate Bose- Fermi exponentiation (sometimes called the plethystic exponential).[9]

Single particle supergravitons in $AdS_4 \times S^7$ transform in an infinite class of representations of the superconformal algebra. The primaries for this spectrum have charges (see [20, 21]) ($\epsilon_0 = \frac{n}{2}, j = 0, h_1 = \frac{n}{2}, h_2 = \frac{n}{2}, h_3 = \frac{n}{2}, h_4 = -\frac{n}{2}$) ($h_1, h_2, h_3$ and h_4 denote SO(8) highest weights in the orthogonal basis; recall Qs here are taken to transform in the vector rather than the spinor of SO(8)) where n runs from 1 to ∞ (we are working with the 'U(N) theory; $n = 1$ would be omitted for the SU(N) theory).

It is not difficult to decompose each of these irreducible representations of the superconformal algebra into representations of the conformal algebra, and thereby compute the partition function and the Index over each of these representations. The necessary decompositions were performed in [20], and we have verified their results independently by means a procedure described in in appendix A. The results are listed in Table 1 below.[10]

It is now a simple matter to compute the Index over single gravitons. The Witten Index for the n^{th} graviton representation (R_n) is given by

$$\begin{aligned} I^W_{R_n} &= \mathrm{Tr}_{\Delta=0}\Big[(-1)^F x^{\epsilon_0+j} y_1^{H_1} y_2^{H_2} y_3^{H_3}\Big] \\ &= \sum_q \frac{(-1)^{2j_q} x^{(\epsilon_0+j)_q} \chi_q^{\mathrm{SO}(6)}(y_1, y_2, y_3)}{1-x^2}, \end{aligned} \tag{2.15}$$

where q runs over all conformal representations with $\Delta = 0$ that appear in the decomposition of R_n in table 1. H_1, H_2, H_3 are the Cartan charges of SO(6) in the 'orthogonal' basis that we always use in this paper. $\chi^{\mathrm{SO}(6)}$, the SO(6) character, may be computed using the Weyl character formula. The full index over single gravitons is

$$I_{sp} = \sum_{n=2}^{\infty} I^W_{R_n} + I^W_{R_1}, \tag{2.16}$$

After some algebra we find

$$\begin{aligned} I_{sp} =& \Big[-x\left(x^2-1\right) y_1 y_2 y_3^2 + \sqrt{x}\sqrt{y_1}\sqrt{y_2}\left(x^3 - y_2 + y_1\left(x^3 y_2 - 1\right)\right) y_3^{3/2} \\ & - x\left(x^2-1\right)\left(y_1+y_2\right)\left(y_1 y_2+1\right) y_3 + \sqrt{x}\sqrt{y_1}\sqrt{y_2}\left(y_2 x^3 + y_1\left(x^3 - y_2\right) - 1\right) \\ & \sqrt{y_3} - x\left(x^2-1\right) y_1 y_2\Big] / \Big[\left(x^2-1\right)\left(\sqrt{x}\sqrt{y_1}\sqrt{y_2} - \sqrt{y_3}\right) \\ & \left(\sqrt{x}\sqrt{y_1}\sqrt{y_3} - \sqrt{y_2}\right)\left(\sqrt{x}\sqrt{y_2}\sqrt{y_3} - \sqrt{y_1}\right)\left(\sqrt{x} - \sqrt{y_1}\sqrt{y_2}\sqrt{y_3}\right)\Big] \end{aligned} \tag{2.17}$$

[9]The index we will calculate is sensitive to $\frac{1}{16}$ BPS states. However, the $\frac{1}{8}$ BPS partition function has been calculated, even at finite N, in [19].

[10]Some of the conformal representations obtained in this decomposition are short (as conformal representations) when n is either 1 or 2; the negative contributions in table 1 represent the charges of the null states, which physically are operators set to zero by the equations of motion. See [22].

range of n	ϵ_0[SO(2)]	SO(3)	SO(8)[orth.(Qs in vector)]	Δ	contribution
$n \geq 1$	$\frac{n}{2}$	0	$\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-n}{2}\right)$	0	+
$n \geq 1$	$\frac{n+1}{2}$	$\frac{1}{2}$	$\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-(n-2)}{2}\right)$	0	+
$n \geq 2$	$\frac{n+2}{2}$	1	$\left(\frac{n}{2}, \frac{n}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2}\right)$	0	+
$n \geq 2$	$\frac{n+3}{2}$	$\frac{3}{2}$	$\left(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2}\right)$	0	+
$n \geq 2$	$\frac{n+4}{2}$	2	$\left(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2}\right)$	1	+
$n \geq 2$	$\frac{n+2}{2}$	0	$\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-(n-4)}{2}\right)$	1	+
$n \geq 3$	$\frac{n+3}{2}$	$\frac{1}{2}$	$\left(\frac{n}{2}, \frac{n}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2}\right)$	1	+
$n \geq 3$	$\frac{n+4}{2}$	1	$\left(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2}\right)$	1	+
$n \geq 3$	$\frac{n+5}{2}$	$\frac{3}{2}$	$\left(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2}\right)$	2	+
$n \geq 4$	$\frac{n+5}{2}$	$\frac{1}{2}$	$\left(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2}\right)$	2	+
$n \geq 4$	$\frac{n+7}{2}$	$\frac{1}{2}$	$\left(\frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2}\right)$	4	+
$n \geq 4$	$\frac{n+6}{2}$	1	$\left(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2}\right)$	3	+
$n \geq 4$	$\frac{n+4}{2}$	0	$\left(\frac{n}{2}, \frac{n}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2}\right)$	2	+
$n \geq 4$	$\frac{n+6}{2}$	0	$\left(\frac{n}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2}\right)$	3	+
$n \geq 4$	$\frac{n+8}{2}$	0	$\left(\frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2}\right)$	6	+
$n = 1$	2	$\frac{1}{2}$	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	1	$-$
$n = 1$	$\frac{5}{2}$	0	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	2	$-$
$n = 2$	3	0	$(1, 1, 0, 0)$	2	$-$
$n = 2$	$\frac{7}{2}$	$\frac{1}{2}$	$(1, 0, 0, 0)$	2	$-$
$n = 2$	4	1	$(0, 0, 0, 0)$	3	$-$

Table 1: d=3 graviton spectrum.

The index over the Fock-space of gravitons may now be obtained from the above single particle index using

$$I_{\text{fock}} = \exp\left(\sum_n \frac{1}{n} I_{sp}(x^n, y_1^n, y_2^n, y_3^n)\right). \tag{2.18}$$

In order to get a feel for this result, let us set $y_i = 1$. The single graviton index reduces to

$$I_{sp} = \frac{2\sqrt{x}\,(2x + \sqrt{x} + 2)}{(\sqrt{x} - 1)^2\,(x + 1)} \tag{2.19}$$

In the high energy limit, $x \equiv e^{-\beta} \to 1$, this expression simplifies to $I_{sp} \approx \frac{20}{\beta^2}$ In this limit the expression for the full Witten Index I_{fock} in (2.18) reduces to,

$$I_{\text{fock}} \approx \exp \frac{20\zeta(3)}{\beta^2} \tag{2.20}$$

It follows that the thermodynamic expectation value of $\epsilon_0 + j$ (which we denote by $E_{\rm av}^{\rm ind}$) is given by

$$E_{\rm av}^{\rm ind} = -\frac{\partial \ln I_{\rm fock}}{\partial \beta} = \frac{40\zeta(3)}{\beta^3}. \tag{2.21}$$

The index 'entropy' defined by

$$I_{\rm fock} = \int dy \exp\{(-\beta y) + S_{\rm ind}(y)\}, \tag{2.22}$$

evaluates to

$$S_{\rm ind}(E) = \frac{60\zeta(3)}{(40\zeta(3))^{\frac{2}{3}}} E^{\frac{2}{3}}. \tag{2.23}$$

It is instructive to compare this result with the relation between entropy and E computed from the supersymmetric partition function, obtained by summing over all supersymmetric states with no $(-1)^F$ — once again in the gravity approximation. The single particle partition function evaluated on the $\Delta = 0$ states with all the other chemical potentials except the one corresponding to $E = \epsilon_0 + j$ set to zero is given by,

$$Z_{sp}(x) = {\rm tr}_{\Delta=0} x^E = \frac{2\sqrt{x}(x+1)\left(x^{5/2} - 2x^2 + 2x^{3/2} + 2x - 3\sqrt{x} + 2\right)}{\left(\sqrt{x}-1\right)^4 (x^2-1)}, \tag{2.24}$$

where once again $x \equiv e^{-\beta}$, with β being the chemical potential corresponding to $E = \epsilon_0 + j$. The bosonic and fermionic contributions to the partition function in (2.24) are respectively given by,

$$Z_{sp}^{\rm bose}(x) = {\rm tr}_{\Delta=0 \ {\rm bosons}} x^E = \frac{-\left(-x^4 + 4x^{7/2} - 6x^3 + x^2 - 4x^{3/2} + 6x - 4\sqrt{x}\right)}{\left(1-\sqrt{x}\right)^5 \left(\sqrt{x}+1\right)(x+1)} \tag{2.25}$$

$$Z_{sp}^{\rm fermi}(x) = {\rm tr}_{\Delta=0 \ {\rm fermions}} x^E = \frac{-\left(-x^4 + x^2 - 4x^{3/2}\right)}{\left(1-\sqrt{x}\right)^5 \left(\sqrt{x}+1\right)(x+1)} \tag{2.26}$$

To obtain the index on the Fock space, we need to multi-particle the partition function above with the correct Bose-Fermi statistics. This leads to

$$Z_{\rm fock} = \exp \sum_n \frac{Z_{sp}^{\rm bose}(x^n) + (-1)^{n+1} Z_{sp}^{\rm fermi}(x^n)}{n}. \tag{2.27}$$

We find, that for $\beta \ll 1$

$$\ln Z_{\rm fock} = \frac{63\zeta(6)}{\beta^5}, \tag{2.28}$$

and a calculation similar to the one done above yields

$$S(E) = \frac{378\zeta(6)}{(315\zeta(6))^{\frac{5}{6}}} E^{\frac{5}{6}}. \tag{2.29}$$

which is the growth of states with energy of a six dimensional gas, an answer that could have been predicted on qualitative grounds. Recall that the theory of the worldvolume

letter	ϵ_0	j	$[h_1, h_2, h_3, h_4]$	$\Delta = \epsilon_0 - j - h_1$
ϕ^a	$\frac{1}{2}$	0	$[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$	0
ψ^a	1	$\frac{1}{2}$	$[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$	0
$\partial\!\!\!/\psi^a = 0$	2	$\frac{1}{2}$	$[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$	1
$\partial^2\phi^a = 0$	$\frac{5}{2}$	0	$[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$	2

Table 2: Spectrum for the single M2 brane.

of the $M2$ brane has 4 supersymmetric transverse fluctuations and one supersymmetric derivative. Bosonic supersymmetric gravitons are in one to one correspondence with 'words' formed by acting on symmetric combinations of these scalars with an arbitrary number of derivatives. Consequently, supersymmetric gravitons are labelled by 5 integers n_i, n_d (the number of occurrences of each of these four scalars $i = 1 \dots 4$ and the derivative n_d) and the energy of these gravitons is $E = \frac{1}{2}(\sum_i n_i) + n_d$. This is the same as the formula for the energy of massless photons in a five spatial dimensional rectangular box, four of whose sides are of length two and whose remaining side is of unit length, explaining the effective six dimensional growth.

We conclude that the growth of states in the effective index entropy is slower than the growth of supersymmetric states in the system; this is a consequence of partial Bose-Fermi cancellations (due to the $(-1)^F$).

2.7 The Index on the worldvolume theory of a single $M2$ brane

We will now compute our index over the worldvolume theory of a single $M2$ brane. For this free theory, the single particle state content is just the representation corresponding to $n = 1$ in table 1 of the previous subsection. This means that it corresponds to the representation of the $d = 3$ superconformal group with the primary having charges $\epsilon_0 = \frac{1}{2}, j = 0$ and SO(8) highest weights (in the convention described above) $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$.

For the reader's convenience, we reproduce the conformal multiplets that appear in this representation in the table above. Physically, these multiplets correspond to the 8 transverse scalars, their fermionic superpartners and the equations of motion for each of these fields.[11]

The Index over these states is

$$
\begin{aligned}
I^{\rm sp}_{M_2}(x, y_i) &= \mathrm{Tr}\left[(-1)^F x^{\epsilon_0 + j} y_1^{H_1} y_2^{H_2} y_3^{H_3}\right] \\
&= \frac{x^{\frac{1}{2}}\left(1 + y_1y_2 + y_1y_3 + y_2y_3\right) - x^{\frac{3}{2}}\left(y_1 + y_2 + y_3 + y_1y_2y_3\right)}{\left(y_1y_2y_3\right)^{\frac{1}{2}}\left(1 - x^2\right)}
\end{aligned}
\tag{2.30}
$$

For simplicity, let us set $y_i \to 1$. Then, we find

$$
I^{\rm sp}_{M_2}(x, y_i = 1) = \frac{4x^{\frac{1}{2}}}{1 + x} \tag{2.31}
$$

[11] Please see [23, 24] and references therein for more details on this worldvolume theory and [25] for some recent work.

Multiparticling this index, to get the index over the Fock space on the M_2 brane, we find that

$$
\begin{aligned}
I_{M_2}(x, y_i = 1) &= \exp \sum_{n \geq 1} \frac{I_{M_2}(x^n, y_i = 1)}{n} \\
&= \left(\prod_{n \geq 0} \frac{1 - x^{2n+\frac{3}{2}}}{1 - x^{2n+\frac{1}{2}}} \right)^4
\end{aligned}
\tag{2.32}
$$

At high temperatures $x \equiv e^{-\beta} \to 1$, the index grows as

$$
I_{M_2}|_{x \to 1, y_i = 1} = \left(\frac{\beta}{2} \right)^{-2} \tag{2.33}
$$

The single particle supersymmetric partition function, obtained by summing over all $\Delta = 0$ single particle states with no $(-1)^F$ is,

$$
\begin{aligned}
Z^{\text{susy,sp}}_{M_2}(x, y_i) &= \text{Tr}_{\Delta=0} \left[x^{\epsilon_0 + j} y_1^{H_1} y_2^{H_2} y_3^{H_3} \right] \\
&= \frac{x^{\frac{1}{2}} \left(1 + y_1 y_2 + y_1 y_3 + y_2 y_3\right) + x^{\frac{3}{2}} \left(y_1 + y_2 + y_3 + y_1 y_2 y_3\right)}{\left(y_1 y_2 y_3\right)^{\frac{1}{2}} \left(1 - x^2\right)}
\end{aligned}
\tag{2.34}
$$

Setting $y_i \to 1$,

$$
Z^{\text{susy,sp}}_{M_2}(x, y_i = 1) = \frac{4x^{\frac{1}{2}}}{1 - x} \tag{2.35}
$$

with individual contributions from bosons and fermions being

$$
\begin{aligned}
Z^{\text{susy,sp,bose}}_{M_2}(x) &= \text{tr}_{\Delta=0 \text{ bosons}} x^E = \frac{4x^{\frac{1}{2}}}{(1 - x^2)} \\
Z^{\text{susy,sp,fermi}}_{M_2}(x) &= \text{tr}_{\Delta=0 \text{ fermions}} x^E = \frac{4x^{\frac{3}{2}}}{(1 - x^2)}
\end{aligned}
\tag{2.36}
$$

Finally, multi-particling this partition function with the appropriate bose-fermi statistics, we find that

$$
Z_{M_2}(x, y_i = 1) = \left(\prod_{n \geq 0} \frac{1 + x^{2n+\frac{3}{2}}}{1 - x^{2n+\frac{1}{2}}} \right)^4 \tag{2.37}
$$

At high temperatures $x \to 1$, the supersymmetric partition function grows as

$$
Z_{M_2}(x \to 1, y_i = 1) \approx \exp \left\{ \frac{\pi^2}{2\beta} \right\} \tag{2.38}
$$

Note, that this partition function grows significantly faster at high temperatures than the index (2.32).

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letter	ϵ_0	j	h	$\Delta = \epsilon_0 - j - h$
ϕ	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$\phi*$	$\frac{1}{2}$	0	$\frac{-1}{2}$	1
ψ	1	$\frac{1}{2}$	$\frac{-1}{2}$	1
$\psi*$	1	$\frac{1}{2}$	$\frac{1}{2}$	0
∂_μ	1	$\{\pm 1, 0\}$	0	$\{0, 2, 1\}$
$\partial_\mu \sigma^\mu \psi = 0$	2	$\frac{1}{2}$	$\frac{-1}{2}$	2
$\partial_\mu \sigma^\mu \psi^* = 0$	2	$\frac{1}{2}$	$\frac{1}{2}$	1
$\partial^2 \phi = 0$	$\frac{5}{2}$	0	$\frac{1}{2}$	2
$\partial^2 \phi^* = 0$	$\frac{5}{2}$	0	$\frac{-1}{2}$	3

Table 3: Spectrum of the free Chern Simons matter theory.

2.8 Index over Chern Simons matter theories

In this subsection, we will calculate the Witten Index described above for a class of the superconformal Chern Simons matter theories recently studied by Gaiotto and Yin [10]. The theories studied by these authors are three dimensional Chern Simons gauge theories coupled to matter fields; we will focus on examples that enjoy invariance under a superalgebra consisting of 4 Qs and 4 Ss (i.e. the R symmetry of these theories is SO(2)). The matter fields, which may thought of as dimensionally reduced $d = 4$ chiral multiplets, carry the only propagating degrees of freedom. The general constructions of Gaiotto and Yin allow the possibility of nonzero superpotentials with a coupling α that flows in the infra-red to a fixed point of order $\frac{1}{k}$ where k is the level of the Chern Simons theory. In our analysis below we will focus on the limit of large k. In this limit, the theory is 'free' and moreover we may treat $\frac{1}{k}$ as a continuous parameter. The arguments above then indicate index that we compute below for the free theory will be invariant under small deformations of $\frac{1}{k}$.

Consider this free conformal 3 dimensional theory on S^2. We are interested in calculating the letter partition function (i.e. the single particle partition function) for the propagating fields which comprise a complex scalar ϕ and its fermionic superpartner ψ. This may be done by enumerating all operators, linear in these fields, modulo those operators that are set to zero by the equations of motion. We will be interested in keeping track of several charges: the energy ϵ_0, SO(3) angular momentum j, SO(2) R-charge h and $\Delta = \epsilon_0 - h - j$ of our states. The table above (which lists these charges) is useful for that purpose The last four lines, with equations of motion count with minus signs in the partition function. The list above comprises two separate irreducible representations of the superconformal algebra. ϕ, ψ and derivatives on these letters make up one representation. The other representation consists of the conjugate fields.

Let the partition functions over these two representations be denoted by z_1 and z_2.

We find

$$
\begin{aligned}
z_1[x,y,t] &= \mathrm{tr}_{\phi,\psi,\dots}(x^{2\epsilon_0}y^{2j}t^h) = \frac{t^{\frac{1}{2}}x(1+x^2)+t^{\frac{-1}{2}}x^2(y+1/y)}{(1-x^2y^2)(1-x^2/y^2)} \\
z_2[x,y,t] &= \mathrm{tr}_{\phi^*,\psi^*,\dots}(x^{2\epsilon_0}y^{2j}t^{2h}) = \frac{t^{\frac{-1}{2}}x(1+x^2)+t^{\frac{1}{2}}x^2(y+1/y)}{(1-x^2y^2)(1-x^2/y^2)}
\end{aligned}
\tag{2.39}
$$

The index (2.11) over single particle states is obtained by setting $t \to 1/x, y \to -1$

$$
\begin{aligned}
I_1[x] &= z_1[x,-1,1/x] = \mathrm{tr}((-1)^F(x)^{2\epsilon_0-h}) = \frac{x^{\frac{1}{2}}}{1-x^2} \\
I_2[x] &= z_2[x,-1,1/x] = \mathrm{tr}((-1)^F x^{2\epsilon_0-h}) = \frac{-x^{\frac{3}{2}}}{1-x^2} \\
I[x] &= I_1[x]+I_2[x] = \frac{x^{\frac{1}{2}}}{1+x}
\end{aligned}
\tag{2.40}
$$

In terms of these quantities, the index of the full theory is given by [26, 27]

$$
I^W = \int DU \exp\left[\sum_{n=1}^{\infty}\sum_m \frac{I(x^n)}{n} Tr_{R_m}(U^n)\right] \tag{2.41}
$$

where m run over the chiral multiplets of the theory, which are taken to transform in the R_m representation of $\mathrm{U}(N)$, and Tr_{R_m} is the trace of the group element in the R_m^{th} representation of $\mathrm{U}(N)$.

In the large N limit the integral over U in (2.41) may be converted into an integral over the eigenvalue distribution of U, $\rho(\theta)$, which, in turn, may be computed via saddle points.[12] The Fourier coefficients of this eigenvalue density function are given by:

$$
\rho_n = \int_{-\pi}^{\pi} \rho(\theta)\cos(n\theta) \tag{2.42}
$$

2.8.1 Adjoint matter

In order to get a feel for this formula, we specialize to a particular choice of matter field content. We consider a theory with c matter fields all in the adjoint representation. In the large N limit the Index is given by

$$
\begin{aligned}
\mathcal{I}(x) &= Tr_{\mathrm{coloursinglets}}(-1)^F x^{2\epsilon_0-h} \\
&= \int d\rho_n \exp\left(-N^2\sum_{n=1}^{\infty}\frac{1}{n}(1-cI[x^n])\rho_n^2\right)
\end{aligned}
\tag{2.43}
$$

The behaviour of this Index as a function of x is dramatically different for $c \leq 2$ and $c \geq 3$. In order to see this note that at any given value of x, the saddle point occurs at $\rho(\theta) = \frac{1}{2\pi}$ i.e $\rho_0 = 1, \rho_n = 0, n > 0$ provided that [26, 27]

$$
1 - cI[x^n] > 0, \forall n \tag{2.44}
$$

[12]Note that $N\rho(\theta)d\theta$ gives the number of eigenvalues between $e^{i\theta}$ and $e^{i(\theta+d\theta)}$ and $\int_{-\pi}^{\pi}\rho(\theta)d\theta = 1$, $\rho(\theta) \geq 0$.

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In this case the saddle point contribution to the Index vanishes; the leading contribution to the integral is then from the Gaussian fluctuations about this saddle point. Under these conditions the logarithm of the Index or the 'free-energy' [13] is then of order 1 in the $\frac{1}{N}$ expansion.

It is easy to check that (2.44) is satisfied at all values of x (which must lie between zero and one in order for (2.11) to be well defined) when $c \leq 2$. On the other hand, if $c \geq 3$ this condition is only met for

$$x < \left(\frac{1}{2}\left(c - \sqrt{c^2-4}\right)\right)^2 \tag{2.45}$$

At this value of x the coefficient of ρ_1^2 in (2.43) switches sign and the saddle point above with a uniform eigenvalue distribution is no longer valid. The new saddle point that dominates this integral above this value of x, has a Gross-Witten type gap in the eigenvalue distribution. The Index undergoes a large N first order phase transition at the critical temperature listed in (2.45). At and above this temperature the 'free-energy' is of order N^2.

Note that $I(1) = \frac{1}{2}$. It follows that the Index is well defined even at strictly infinite temperature This is unlike the logarithm of the actual partition function of the same theory, whose $x \to 1$ limit scales like $N^2/(1-x)^2$ as $x \to 1$ (for all values of c) reflecting the T^2 dependence of a 2+1 dimensional field theory. This difference between the high temperature limits of the Index and the partition function reflects the large cancellations of supersymmetric states in their contribution to the Index.

2.8.2 Fundamental matter

As another special example, let us consider a theory whose N_f matter fields all transform in the fundamental representation of U(N). We take the Veneziano limit: $N_c \to \infty, c = \frac{N_f}{N_c}$ fixed. The index for the theory is now given by

$$\begin{aligned}\mathcal{I}(x) &= Tr_{\text{coloursinglets}}(-1)^F x^{2\epsilon_0 - h} \\ &= \int d\rho_n \exp(-N^2 \sum_{n=1}^{\infty} \frac{(\rho_n - cI[x^n])^2 - c^2 I[x^n]^2}{n})\end{aligned} \tag{2.46}$$

At low temperatures the integral in (2.46) is dominated by the saddle point

$$\rho_n = cI(x^n) \tag{2.47}$$

As the temperature is raised the integral in (2.46) undergoes a Gross-Witten type phase transition when c is large enough. This is easiest to appreciate in the limit $c \gg 1$. In this limit $\rho_1 = \frac{1}{2}$ in the low temperature phase when at $x \approx \frac{1}{4c^2}$, and $\rho_n = \frac{1}{2^n c^{n-1}} \ll 1$. At approximately this value of x the low temperature eigenvalue distribution $\rho(\theta)$ formally turns negative at $\theta = \pi$. This is physically unacceptable (as an eigenvalue density is, by definition, intrinsically positive). In actual fact the system undergoes a phase transition at this value of x. At large c this phase transition is very similar to the one described by Gross

[13] We use this term somewhat loosely, since we are referring here to an index and not a partition function

and Witen in [28] and in a more closely related context by [29]. The high temperature eigenvalue distribution is 'gapped' i.e. it has support on only a subset (centered about zero) of the interval $(-\pi, \pi)$.

For this phase transition to occur, we need $c \geq 3$. To arrive at this result, we notice that the distribution (2.47) implies

$$\lim_{x\to 1-} \rho(\pi) = \lim_{x\to 1-} \rho(-\pi) = \frac{1}{\pi}\left(\frac{1}{2} - \frac{c}{4}\right) \tag{2.48}$$

So, for $c \geq 3$, $\rho(\pi)$ would always turn negative for some value of x. Beyond this temperature the saddle point (2.47) is no longer valid.

3. d=6

3.1 The superconformal algebra and its unitary representations

The bosonic subalgebra of the $d = 6$ superconformal algebra is $\mathrm{SO}(6,2) \otimes \mathrm{Sp}(2n)$ (the conformal algebra times the R symmetry algebra). The anticommuting generators in this algebra may be divided into the generators of supersymmetry (Q) and the generators of superconformal symmetries (S). Supersymmetry generators transform in the fundamental representation of the R-symmetry group $\mathrm{Sp}(2n)$,[14] have charge half under dilatations (the SO(2) factor of the compact $\mathrm{SO}(6) \otimes \mathrm{SO}(2) \in \mathrm{SO}(6,2)$) and are chiral spinors under the SO(6) factor of the same decomposition. Superconformal generators $S_i^\mu = (Q^i_\mu)^\dagger$ transform in the anti-chiral spinor representation of SO(6), have scaling dimension (dilatation charge) $(-\frac{1}{2})$, and also transform in the anti-fundamental representation of the R-symmetry group. The charges of these generators are given in more detail in appendix B. In our notation for supersymmetry generators i is an SO(6) spinor index while μ is an R symmetry vector index.

The commutation relations for this superalgebra are described in detail in [4]. As usual, the anticommutator between two supersymmetries is proportional to momentum times an R symmetry delta function, and the anticommutator between two superconformal generators is obtained by taking the Hermitian conjugate of these relations. The most interesting relationship in the algebra is the anticommutator between Q and S. Schematically

$$\{S_i^\mu, Q_\nu^j\} \sim \delta_\nu^\mu T_i^j - \delta_i^j M_\nu^\mu$$

Here T^{ij} are the $\mathrm{U}(4) \sim \mathrm{SO}(6) \times \mathrm{SO}(2)$ generators and $M_{\mu\nu}$ are the $\mathrm{Sp}(2n)$ generators. The energy ϵ_0, SO(6) highest weight (denoted by h_1, h_2 and h_3 in the orthogonal basis[15]) and the R-symmetry highest weights $(k, k_1 \ldots, k_{(n-1)})$ of primary states form a complete set of labels for the representation in question. We use a non-standard normalization for the R-symmetry weights. In particular,

$$k = \frac{k^o}{2}, \quad k_i = \frac{k_i^o}{2} \tag{3.1}$$

[14] With our conventions, $\mathrm{Sp}(2n)$ is of rank n. $\mathrm{Sp}(2) = \mathrm{SO}(3)$ and $\mathrm{Sp}(4) = \mathrm{SO}(5)$.

[15] h_i are eigenvalues under rotations in orthogonal 2 planes in R^n. Thus, for instance, $\{h_i\} = (1,0,0)$ in the vector representation. They are either integer or half integer and satisfy the constraint $h_1 \geq h_2 \geq |h_3| \geq 0$.

Here $[k^o, k^o_i]$ are the highest weights of Sp($2n$) in the orthogonal basis.[16] As we have noted above, at the level of the algebra, SO(2) $\times$ SO(6) $\sim$ U(4). We will sometimes find it convenient to label primaries by eigenvalues c_i under the generators $T^i_i \equiv T_i$ of U(4)[17] rather than by the energy and SO(6) weights. For any highest weight (c_1, c_2, c_3, c_4) the eigenvalues satisfy $c_1 \geq c_2 \geq c_3 \geq c_4 \geq 0$ and c_i s are always integers. For future reference we note the change of basis between the Cartan elements $\epsilon_0, h_1, h_2, h_3$ (the energy and 3 orthogonal SO(6) Cartan generators) and T_1, T_2, T_3, T_4:

$$\begin{aligned} \epsilon_0 &= \frac{1}{2}(T_1 + T_2 + T_3 + T_4) \\ h_1 &= \frac{1}{2}(T_1 + T_2 - T_3 - T_4) \\ h_2 &= \frac{1}{2}(T_1 - T_2 + T_3 - T_4) \\ h_3 &= \frac{1}{2}(T_1 - T_2 - T_3 + T_4) \end{aligned} \tag{3.2}$$

As in the case of the $d = 3$ algebra, any irreducible representation of the superconformal algebra may be decomposed into a finite number of distinct irreducible representations of the conformal algebra. The latter are labeled by their own conformal primary states, which have a definite lowest energy and transform in a given irreducible representation of SO(6).

We will now analyse the constraints imposed by unitarity on the quantum numbers of primary states; for this purpose we will find it convenient to use the U(4) labeling of primaries introduced above. Let Q^i_μ $i = 1, \cdots, 4.$ and $\mu = \pm 1, \cdots, \pm n$ denote the supersymmetry whose charge under U(4) Cartan T_j are δ^i_j and under the R-symmetry Cartan M_ν is (sign ofμ)$\times \delta^\nu_{|\mu|}$. The superconformal generators are $S^\mu_i = (Q^i_\mu)^\dagger$ and therefore they have the same charges as Q^i_μ but with opposite sign.

3.2 Norms and null states

In this subsection we study unitarity restrictions (and the resultant structure of null states) of representations of the superconformal algebra. This analysis turns out to be a little more intricate than its $d = 3$ counterpart.

As we have seen above, states in the same representation of the superconformal algebra do not all have the same norm. However states that lie within the same representation of the maximal compact subgroup of the algebra, U(4) $\times$ Sp($2n$), do have the same norm. Consequently, in order to examine the constraints from unitarity, we need only examine one state per representation of this compact subalgebra.

In order to study the restrictions imposed by unitarity at level ℓ we should, in principle, study all states obtained by acting with the tensor product of an arbitrary combination of ℓ supersymmetries on the set of primary states of an irreducible representation of the superconformal algebra. This set of states may be Clebsh Gordan decomposed into a sum of irreducible representations of U(4) $\times$ Sp($2n$); and we should compute the norm of at least

[16] In the orthogonal basis, the Cartans of Sp($2n$) are $2n \times 2n$ matrices with elements diag($i\sigma_2, 0, 0 \ldots$), diag($0, i\sigma_2, 0, 0, \ldots$), $\ldots$, where each 0 is shorthand for a 2×2 matrix.

[17] In the defining representation of U(4) $(T_i)^a_b = \delta^a_i \delta^i_b$.

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one state in each of these representations, and ensure its positivity in order to guarantee unitarity. However this problem is significantly simplified by the observation that the most stringent condition on unitarity occurs in those states that transform in the 'largest' Sp($2n$) [5]. Now it is easy to construct a state in the largest Sp($2n$) representation: one simply acts on those primary states that are Sp($2n$) highest weight with ℓ Sp($2n$) highest weight supersymmetries, i.e. supersymmetries of the form Q^i_1. This prescription completely fixes the Sp($2n$) quantum numbers of the states we will study in this section. All that remains is to study the decomposition of all such states into irreducible representations of U(4) and to compute the norm of one state in each of these representations.

The decomposition of the states of interest into U(4) representations at level ℓ is easily performed using Young tableaux techniques. The set of U(4) tableaux for representations of the descendants is obtained by adding ℓ boxes to the tableaux of the primary in all possible ways that give rise to a legal tableaux, subject to the restriction that no two 'new' boxes occur on the same row (this restriction is forced on us by the antisymmetry of the Q^i_1 operators). Note, that in this decomposition, no representation occurs more than once.[18]

It is not too difficult to find an explicit formula for the highest weight states of each of these representations. Let us define the operators $\left(A^i = \sum_{j=1}^{i} Q^j_1 \Upsilon^i_j\right)$ $i = 1, \cdots, 4$ where Υ^i_j are functions of the U(4) generators defined by

$$\begin{aligned}
\Upsilon^j_j =& \text{Identity} \quad \text{(no sum over j)} \\
\Upsilon^4_1 =& -\left[T^2_1 T^3_2 T^4_3 \left(\frac{(T_3-T_4+1)(T_2-T_4+2)}{(T_3-T_4)(T_2-T_4+1)}\right)\right. \\
& \left. - T^3_2 T^4_3 T^2_1 \left(\frac{T_3-T_4+1}{T_3-T_4}\right) - T^4_3 T^2_1 T^3_2 \left(\frac{T_2-T_4+2}{T_2-T_4+1}\right) + T^4_3 T^3_2 T^2_1\right] \left(\frac{1}{T_1-T_4+2}\right) \\
\Upsilon^4_2 =& -\left(T^4_3 T^3_2 - T^3_2 T^4_3 \left(\frac{T_3-T_4+1}{T_3-T_4}\right)\right)\left(\frac{1}{T_2-T_4+1}\right) \\
\Upsilon^4_3 =& -T^4_3 \left(\frac{1}{T_3-T_4}\right) \\
\Upsilon^3_1 =& -\left(T^3_2 T^2_1 - T^2_1 T^3_2 \left(\frac{T_2-T_3+1}{T_2-T_3}\right)\right)\left(\frac{1}{T_1-T_3+1}\right) \\
\Upsilon^3_2 =& -T^3_2 \left(\frac{1}{T_2-T_3}\right) \\
\Upsilon^2_1 =& -T^2_1 \left(\frac{1}{T_1-T_2}\right)
\end{aligned} \tag{3.3}$$

The operators A^i have been determined to have the following property: when acting on a highest weight state $|\psi\rangle$ of U(4) with quantum numbers (c_1, c_2, c_3, c_4), $A^i|\psi\rangle$ is another highest weight state of U(4) with quantum numbers $(c^i_1, c^i_2, c^i_3, c^i_4)$ where $c^i_j = c_j + \delta^j_i$,

[18] For a generic primary tableaux the number of representations obtained at level ℓ is $\binom{4}{\ell}$ corresponding to the choice of which rows the new boxes are appended to. If the U(4) highest weights of the primary are c_1, c_2, c_3, c_4, the representation obtained by appending new boxes to the rows R^{i_1}, R^{i_ℓ} has highest weights $c_{i_1} \dots c_{i_\ell}$ increased by one, while all other weights are unchanged.

whenever it is well defined. The last condition (being well defined) is met if and only if the weights of $|\psi\rangle$ obey the inequality $c_i < c_{i-1}$.[19]

Let $|\psi\rangle$ denote the primary state that is a U(4) highest weight. It follows that the states $A^{i_1} \dots A^{i_\ell}|\psi\rangle$ is the highest weight state in the representation with additional boxes in the rows $i_1 \dots i_\ell$ described above. We will now study the norm of these states.

It is not difficult to explicitly verify that (when this state is well defined)

$$|A^i|\psi\rangle|^2 \propto (c_i - 2k - i + 1) \equiv B_i(c_i, k) \tag{3.4}$$

More generally, it is also true that

$$|\prod_{m=1}^{l} A^{i_m}|\psi\rangle|^2 \propto \prod_{m=1}^{l} B_i(c_{i_m}, k) \tag{3.5}$$

where the proportionality factor in (3.5) is a function of the the SU(3) weights $c_i - c_j$ of the representation but is independent of the energy.[20] In order to see this note that different states of the form (3.5), obtained by interchanging the order of the A^{i_m} operators, are each proportional to the highest weight state of a given representation. Now no U(4) representation occurs more than once in the tensor product of supersymmetry generators with the primary, these representations are proportional to each other. As the commutator of A^i operators is independent of energy, it follows that the proportionality factor between these states is also independent of energy.

Now the norm of the state in (3.5) clearly has a factor of $B_{i_l}(c_i)$ in it. However upon interchanging the order of the A^i factors, the same result is true for B_{i_m} for each of $m = 1$ to l. The norm of a state at level ℓ is a polynomial of degree ℓ in the energy of the state. It follows that the full energy dependence of the norm of this state is given as in (3.5); the proportionality factor in that equation is a function only of SU(3) weights and is independent of energy.

The proof presented above, strictly speaking, applies only when each of the operators A^{i_m} has well defined action on $|\psi\rangle$. However, as the algebra involved in computing (3.5) is smooth (it does not care about the values of c_i provided only that the state on the l.h.s. of (3.5) is well defined), and so the result (3.5) continues to apply, whenever the state whose norm is being computed is well defined.

The unitarity restrictions and short representations of this superconformal algebra now follow almost immediately from (3.5). First consider the generic case representation where $(c_1 > c_2 > c_3 > c_4)$. All states listed in (3.5) are well defined in this case and it follows $c_4 - 3 - 2k \geq 0$ is necessary and sufficient for unitarity. Representations that saturate this bound are short; the zero norm primary state is

$$|Z_4\rangle = A^4|h.w\rangle \tag{3.6}$$

[19]This is rather intuitive; when this condition is not met, the set $(c_1^i, c_2^i, c_3^i, c_4^i)$ do not constitute a valid set of labels for an irreducible representation of U(4).

[20]More precisely, the proportionality factor is a function of the c_i that is invariant under a uniform constant shift of each c_i.

consistent with the result of [7].

The state (3.5) is not well defined when $c_3 = c_4$. However even in this case the state $\left(A^4 A^3\right)|\psi\rangle$ is well defined provided $c_2 \neq c_3$. The norm of this state is proportional to $B_4 \times B_3$. A little thought shows that the necessary and sufficient condition for unitarity is either $B_4 \geq 0$ (this is (3.6)) or that $B_3 = 0$. In the later case the representation is short, and its level one zero norm primary is $A^3|\psi\rangle$. On the other hand when $B_4 = 0$ the representation is also short. It's zero norm primary occurs at level 2 and is $\left(A^4 A^3\right)|\psi\rangle$.

It is clear that this pattern generalizes simply. If $c_4 = c_3 = c_2$ but $c_2 \neq c_1$ then the necessary and sufficient condition for unitarity is either $B_4 \geq 0$ or $B_3 = 0$ or $B_2 = 0$. When $B_2 = 0$ the zero norm primary occurs at level one and is given by $A^2|\psi\rangle$. When $B_3 = 0$ the zero norm primary occurs at level 2 and is given by $\left(A^3 A^2\right)|\psi\rangle$. When $B_4 = 0$ the zero norm primary occurs at level 3 and is given by $\left(A^4 A^3 A^2\right)|\psi\rangle$.

Finally when $c_4 = c_3 = c_2 = c_1$ the necessary and sufficient condition for unitarity is either $B_4 \geq 0$ or $B_3 = 0$ or $B_2 = 0$ or $B_1 = 0$. When $B_1 = 0$ the level one primary is given by $A^1|\psi\rangle$. When $B_2 = 0$ the level two primary is given by $\left(A^2 A^1\right)|\psi\rangle$. When $B_3 = 0$ the level three primary is given by $\left(A^3 A^2 A^1\right)|\psi\rangle$. When $B_4 = 0$ the level four primary is given by $\left(A^4 A^3 A^2 A^1\right)|\psi\rangle$.

We may translate the analysis of zero norm states above into SO(2) × SO(6) notation by using the transformations of (3.2). This yields the result that representations are short if the energy ϵ_0 and SO(6) weights h_i satisfy one of the following conditions (see [4, 5])

$$\begin{aligned}
\epsilon_0 =& h_1 + h_2 - h_3 + 4k + 6, \quad \text{when } \mathrm{h}_1 \geq \mathrm{h}_2 \geq |\mathrm{h}_3|. \\
\epsilon_0 =& h_1 + 4k + 4, \quad \text{when } \mathrm{h}_1 \geq \mathrm{h}_2 \text{ and } \mathrm{h}_2 = \mathrm{h}_3. \\
\epsilon_0 =& h_1 + 4k + 2, \quad \text{when } \mathrm{h}_1 = \mathrm{h}_2 = \mathrm{h}_3 \neq 0. \\
\epsilon_0 =& 4k, \quad \text{when } \mathrm{h}_1 = \mathrm{h}_2 = \mathrm{h}_3 = 0.
\end{aligned} \tag{3.7}$$

The last three conditions give isolated short representations.

3.3 Null vectors and character decomposition of a long representation at the unitarity threshold

As discussed in the previous subsection, just like $d = 3$ the short representations of $d = 6$ super-conformal algebra can be broadly classified into two types, the *regular* ones and the *isolated* ones. However unlike $d = 3$ here the isolated short representations are of three kinds as we describe below. The energy of a regular short representations is given by $\epsilon_0 = h_1 + h_2 - h_3 + 4k + 6$. The null states of this representation also transform in an irreducible representation of the algebra; for $h_1 > h_2$ and $h_2 - \frac{1}{2} > |h_3 - \frac{1}{2}|$ the highest weights of the primary at the head of this (null) irreducible representation (which occurs at level 1) are given in terms of the highest weight of the representation by $\epsilon_0' = \epsilon_0 + \frac{1}{2}$, $h_1' = h_1 - \frac{1}{2}$, $h_2' = h_2 - \frac{1}{2}$, $h_3' = h_3 + \frac{1}{2}$, $k' = k + \frac{1}{2}$, $k_i' = k_i$ (where $i = 1, 2, \ldots, (n-1)$) and k, k_i are half the weights of the R-symmetry group $Sp(2n)$ in the orthogonal basis as defined in subsection (§§ 3.1). Note that $\epsilon_0' - h_1' - h_2' + h_3' - 4k' - 6 = \epsilon_0 - h_1 - h_2 + h_3 - 4k - 6 = 0$, so that the null states also transform in a regular short representation. As union of the ordinary and null state of such short representations is identical to the state content of a long representation at the edge of the unitarity bound, we conclude that,

$$\begin{aligned}\lim_{\delta\to 0}\chi[h_1{+}h_2-h_3+4k+6+\delta,h_1,h_2,h_3,k,k_i]&\\ =\chi[h_1+h_2-h_3+4k+6,h_1,h_2,h_3,k,k_i]&\\ +\chi\left[h_1+h_2-h_3+4k+\frac{13}{2},h_1-\frac{1}{2},h_2-\frac{1}{2},h_3+\frac{1}{2},k+\frac{1}{2},k_i\right],&\\ (\text{with } \mathrm{h}_1>\mathrm{h}_2>|\mathrm{h}_3|\geq 0).&\end{aligned}\tag{3.8}$$

where $\chi(\epsilon_0,h_1,h_2,h_3,k,k_i)$ denotes the character of the irreducible representation of superconformal algebra with energy ϵ_0, SO(6) highest weight (h_1,h_2,h_3) and Sp($2n$) highest weight (k,k_i).

On the other hand, when $h_1>h_2=h_3(=h$, say) the null states of the regular short representation occur at level 2 and are labelled by a primary with highest weights $\epsilon_0'=\epsilon_0+1,\quad h_1'=h_1-1,\quad h_2'=h_2=h,\quad h_3'=h_3=h,\quad k'=k+1,\quad k_i'=k_i$, where ϵ_0,h_i,k,k_i refer to the highest weights of the original representation. Note in particular that $h_2'=h_3'$ and $\epsilon_0'-h_1'-4k'-4=\epsilon_0-h_1-h_2+h_3-4k-6=0$. It follows that the null states of this representation transforms in an isolated short representation and we conclude,

$$\begin{aligned}\lim_{\delta\to 0}\chi[h_1+4k+6+\delta,h_1,h,h,k,k_i]=&\chi[h_1+4k+6,h_1,h,h,k,k_i]\\ &+\chi[h_1+4k+7,h_1-1,h,h,k+1,k_i]\\ &(\text{with } \mathrm{h}_1>\mathrm{h}_2=\mathrm{h}_3=\mathrm{h}\geq 0).\end{aligned}\tag{3.9}$$

As we have discussed earlier isolated short representations are separated from all other representations with the same SO(6) and Sp($2n$) quantum numbers by a gap in energy. Hence it is not possible to *approach* such a representation with long representations; consequently we have no equivalent of (3.9) at energies equal to $h_1+4k+7+\delta$.

Similarly when $h_1=h_2=h_3(=h\neq 0)$ the null states of the regular representation occur at level 3 and are labelled by a primary with highest weights $\epsilon_0'=\epsilon_0+\frac{3}{2},\quad h_1'=h-\frac{1}{2},\quad h_2'=h-\frac{1}{2},\quad h_3'=h-\frac{1}{2},\quad k'=k+\frac{3}{2}$. Note in particular that $h_1'=h_2'=h_3'$ and $\epsilon_0'-h_1'-4k'-2=\epsilon_0-h_1-4k-6=0$. Consequently the null states of this representation transforms in an isolated short representation, and we conclude,

$$\begin{aligned}\lim_{\delta\to 0}\chi[h+4k+6+\delta,h,h,h,k,k_i]=&\chi[h+4k+6,h,h,h,k,k_i]\\ &+\chi\left[h+4k+\frac{15}{2},h-\frac{1}{2},h-\frac{1}{2},h-\frac{1}{2},k+\frac{3}{2},k_i\right].\\ &(\text{with } \mathrm{h}_1=\mathrm{h}_2=\mathrm{h}_3=\mathrm{h}>0)\end{aligned}\tag{3.10}$$

As explained above, we have no equivalent of (3.10) at energies equal to $h+4k+\frac{15}{2}+\delta$ which corresponds to the unitarity bound for an isolated short representation.

Finally when $h_1=h_2=h_3=0$ the null states of the regular representation occur at level 4 and are labelled by primary with highest weights $\epsilon_0'=\epsilon_0+2,\quad h_1'=h_1=$

notation for rep.	ϵ_0	SO(6) highest weight	Sp(2n) highest weight	nature of rep
$c(h_1,h_2,h_3,k,k_i)$ (with $h_1 \geq h_2 \geq \|h_3\|$ and $k \geq 0$)	$h_1+h_2-h_3+4k+6$	(h_1,h_2,h_3)	(k,k_i)	regular short
$c(h_1,h-\frac{1}{2},h+\frac{1}{2},k,k_i)$ (with $h_1 \geq h+\frac{1}{2}$ $h \geq 0$ and $k \geq -\frac{1}{2}$)	$h_1+4k+\frac{11}{2}$	$(h_1-\frac{1}{2},h,h)$	$(k+\frac{1}{2},k_i)$	isolated short
$c(h,h,h+1,k,k_i)$ (with $h \geq 0$ and $k \geq -1$)	$h+4k+6$	(h,h,h)	$(k+1,k_i)$	isolated short
$c(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},k,k_i)$ (with $k \geq -\frac{3}{2}$	$4k+6$	$(0,0,0)$	$(k+\frac{3}{2},k_i)$	isolated short

Table 4: Notations for short representations.

0, $h_2' = h_2 = 0$, $h_3' = h_3 = 0$, $k' = k+2$, $k_i' = k_i$. Note in particular that in this case $h_1' = h_2' = h_3' = 0$ and $\epsilon_0' - 4k' = \epsilon_0 - 4k - 6 = 0$. Therefore the null states of this representation transform in an isolated short representation and we conclude,

$$\lim_{\delta\to 0} \chi[4k+6+\delta,0,0,0,k,k_i] = \chi[4k+6,0,0,0,k,k-i]+\chi[4k+8,0,0,0,k+2,k_i]. \qquad (\text{with } \mathrm{h}_1=\mathrm{h}_2=\mathrm{h}_3=0) \tag{3.11}$$

There is no equivalent of (3.11) at energies equal to $4k+6+\delta$.

As in the previous section, the analysis of the character formulae above and the definition of indices is much simplified by the introduction of some additional notation. Given a short representation we will use the notation $c(h_1,h_2,h_3,k,k_i)$ to refer to this representation where the relationship between the numbers h_i,k,k_i and the highest weights of the representation in question is defined in table 4.

3.4 Indices

As in the $d=3$ case, we define an index for $d=6$ as any linear combination of the multiplicities of short representations that evaluates to zero on every collection of representations that appear on the r.h.s. of (3.8), (3.9), (3.10),and (3.11) so that it is invariant under any deformation of superconformal field theory under which the spectrum evolves continuously. We now proceed to list all of these indices,

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1. The simplest indices are given by the multiplicities of short representations in the spectrum that never appear on the r.h.s. of (3.8), (3.9), (3.10),and (3.11) (for any values of the quantum numbers on the l.h.s. of those equations). All such representations are easy to list; they are

   - $c(h_1, h-\frac{1}{2}, h+\frac{1}{2}, k, k_i)$ for all $h_1 \geq h+\frac{1}{2}$, $h \geq 0$ and $k - k_1 = -\frac{1}{2}, 0$.
   - $c(h, h, h+1, k, k_i)$ for all $h \geq 0$ and $k - k_1 = -1, -\frac{1}{2}, 0$
   - $c(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, k, k_i)$ for $k - k_1 = -\frac{3}{2}, -1, -\frac{1}{2}, 0$

   In all the above cases we must consider all the possible values of the set $k_i, i = 1 \ldots n-1$. This means $k_1 \geq k_2 \geq \ldots \geq k_{n-1} \geq 0$ and the k_i may each be integers or half integers.

2. We can also construct indices from linear combinations of the multiplicities of representations that do appear on the r.h.s. of (3.8), (3.9), (3.10),and (3.11). The complete list of such linear combinations is given by,

$$I_{M_1,M_2,M_3,\{k_i\}} = \sum_{p=M_3-1}^{2(M_1-k_1)} (-1)^{p+1} n\left\{c\left(M_2 + \frac{p}{2}, \frac{p}{2}, M_3 - \frac{p}{2}, M_1 - \frac{p}{2}, k_i\right)\right\}, \tag{3.12}$$

   where $n\{R\}$ denotes the number of representations of type R and the Index labels M_1, M_2 and M_3 are respectively the values of h_2+k, h_1-h_2 and $M_3 = h_2+h_3$ for the regular representations that appears in the above sum. Here M_2 and M_3 are integers greater than or equal to zero and M_1 is an integer or half integer with $M_1 \geq \frac{M_3}{2} + k_1$.

3.5 Minimally BPS states: distinguished supercharge and commuting superalgebra

Consider the special Q with charges ($h_1 = -\frac{1}{2}, h_2 = -\frac{1}{2}, h_3 = \frac{1}{2}, k = \frac{1}{2}, \epsilon_0 = \frac{1}{2}$). Let $S = Q^\dagger$; it is then easily verified that,

$$2\{S, Q\} \equiv \Delta = \epsilon_0 - (h_1 + h_2 - h_3 + 4k) \tag{3.13}$$

Just as in $d = 3$, we shall define a partition function over states annihilated by Q. Again all such states transform in an irreducible representation of the subalgebra of the superconformal algebra that commutes with Q, S and hence Δ. This subalgebra is easily determined to be the supergroup $D(3, \frac{n-2}{2})$ (see [4]).

The bosonic subgroup of this commuting superalgebra is $\mathrm{SU}(3,1) \otimes \mathrm{Sp}(n-2)$. The usual Cartan charges of $\mathrm{SU}(3,1)$ and the Cartan charges of $\mathrm{Sp}(n-2)$ are given in terms of the Cartan elements of the full superconformal algebra by,

$$E = 3\epsilon_0 + h_1 + h_2 - h_3; H_1 = h_1 - h_2; H_2 = h_2 + h_3; K_i = k_{i+1}, \tag{3.14}$$

where E is the U(1) Cartan, (H_1, H_2) are the SU(3) Cartans (in the Dynkin basis) and K_i are the Sp$(n-2)$ Cartans (in the orthogonal basis).[21]

3.6 A trace formula for the general index and its character decomposition

As in the case of $d = 3$, we define the Witten index as,

$$I^W = Tr_R[(-1)^F \exp\left(-\zeta\Delta + \mu G\right)], \tag{3.15}$$

Where the trace is evaluated over any Hilbert space that hosts a representation of the $d = 6$ superconformal algebra. Here F is the fermion number operator; by the spin statistics theorem, in any quantum field theory we take $F = 2h_2$. G is any element of the subalgebra that commutes with the set set $\{Q, S, \Delta\}$; by a similarity transformation , G may always be rotated in to a linear combination of the subalgebra Cartan generators.

The Witten index (3.15) receives contributions only from the states that are annihilated by both Q and S (all other states yields contribution that cancel in pairs) and, hence, have $\Delta = 0$. So it is independent of ζ. The usual arguments [6] also ensure that I^W is also an index and hence it should be possible to expand I^W as a linear combination of the indices defined in the previous section. In fact is easy to check that for any representation A (reducible or irreducible) of the $d = 6$ superconformal algebra,

$$\begin{aligned}
I^{wi}(A) = & \sum_{M_1,M_2,M_3,\{k_i\}} I_{M_1,M_2,M_3}\chi_{\text{sub}}(M_2, M_3, k_i, 4(M_2 - M_3) + 12M_1 + 24) \\
+ & \sum_{\{k_i\},k-k_1=-\frac{3}{2},-1,-\frac{1}{2},0} n\Big\{c\Big(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, k, k_i\Big)\Big\}\chi_{\text{sub}}(0, 0, k_i, 12k + 18) \\
+ & \sum_{\{k_i\},h\geq 0,k-k_1=-1,-\frac{1}{2},0} (-1)^{2h+1} n\{c(h, h, h+1, k, k_i)\}\chi_{\text{sub}}(0, 2h+1, k_i, 4h + 12k + 20) \\
+ & \sum_{\{k_i\},h_1,h(h_1\geq h\geq 0),k-k_1=-\frac{1}{2},0} (-1)^{2h} n\{c(h_1, h, h+1, k, k_i)\}\chi_{\text{sub}}(h_1-h, 2h+1, k_i, 4h_1+12k+20).
\end{aligned} \tag{3.16}$$

where $\chi_{\text{sub}}(H_1, H_2, K_i, E)$ is the supercharacter of the representation with highest weights H_1, H_2, K_i, E as defined in (3.14). In the first sum in (3.16) M_2 and M_3 run over integers greater than or equal to zero and M_1 runs over integers or half integers with $M_1 \geq \frac{M_3}{2} + k_1$. Also the set $\{k_i\}$ runs over integer and half integer values satisfying the condition $k_1 \geq k_2 \cdots \geq k_n$. In order to obtain (3.16) we have used,

$$\begin{aligned}
I^{wi}[(c(h_1, h_2, h_3, k, k_i)(\text{with } \mathrm{h}_1 \geq \mathrm{h}_2 \geq |\mathrm{h}_3| \text{ and } \mathrm{k} \geq 0)] = & \\
(-1)^{2h_2+1}\chi_{\text{sub}}(h_1 - h_2, h_2 + h_3, k_i, 4(h_1 + h_2 - h_3) + 12k + 24). &
\end{aligned} \tag{3.17}$$

[21] Specifically the Cartans H_1 and H_2 are the following 3×3 SU(3) matrices,

$$H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$I^{wi}\left[c(h_1,h,h+1,k,k_i)\left(\text{with } h_1\geq h\geq 0 \text{ and k} \geq -\frac{1}{2}\right)\right] = \\ (-1)^{2h+1}\chi_{\rm sub}(h_1-h,2h+1,k_i,4h_1+12k+20). \tag{3.18}$$

$$I^{wi}[(c(h,h,h+1,k,k_i)(\text{with h} \geq 0 \text{ and k} \geq -1)] = \\ (-1)^{2h+1}\chi_{\rm sub}(0,2h+1,k_i,4h+12k+20). \tag{3.19}$$

$$I^{wi}\left[c\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},k,k_i\right)\left(\text{with k}\geq -\frac{3}{2}\right)\right] = \chi_{\rm sub}(0,0,\rm k_i,12k+18). \tag{3.20}$$

Equations (3.17)-(3.20) follow from the observation that the set of $\Delta = 0$ states (the only states that contribute to the Witten index) in any short representation of the superconformal algebra transform in a single representation of the commuting super subalgebra. The quantum numbers of these representations of the subalgebra are easily determined, given the quantum numbers of the parent short representation. In the case of regular short representations, a primary of the subalgebra representation (in which the $\Delta = 0$ states transform)is obtained by the acting on the highest weight primary of the full representation (which turns out to have $\Delta = 6$) with supercharges Q_1, Q_2 and Q_3 with quantum numbers $(h_1 = \frac{1}{2}, h_2 = \frac{1}{2}, h_3 = \frac{1}{2}, k = \frac{1}{2}, k_i = 0, \epsilon_0 = \frac{1}{2})$, $(h_1 = \frac{1}{2}, h_2 = -\frac{1}{2}, h_3 = -\frac{1}{2}, k = \frac{1}{2}, k_i = 0, \epsilon_0 = \frac{1}{2})$ and $(h_1 = -\frac{1}{2}, h_2 = \frac{1}{2}, h_3 = -\frac{1}{2}, k = \frac{1}{2}, k_i = 0, \epsilon_0 = \frac{1}{2})$ respectively, all of which has $\Delta = -2$. The Witten index evaluated over these representations in terms of the supercharacter of the subgroup is given by (3.17).

In the case of isolated representations the highest weight primary of the full representation turns out to have $\Delta = 4, 2$ and 0; for the $\Delta = 4$ case the primary of the subalgebra is obtained by the action of Q_1 and Q_2 on the primary of the full superconformal algebra, and for $\Delta = 2$ case it is obtained by the action of Q_1. The highest weight of an isolated superconformal short which itself has $\Delta = 0$ is also a primary of the commuting subalgebra. The Witten index evaluated over these representations in terms of the supercharacter of the subgroup is given by (3.18), (3.19) and (3.20).

Note that every index that appears in the list of subsection §§ 3.4 appears as the coefficient of a distinct subalgebra supercharacter in (3.16). As supercharacters of distinct irreducible representations are linearly independent, it follows that knowledge of I^W is sufficient to reconstruct all superconformal indices of the algebra. In this sense (3.16) is the most general index that can be constructed from the superconformal algebra alone.

3.7 The Index over M theory multi gravitons in $AdS_7 \times S^4$

We now compute the Witten Index defined for the for the world volume theory of the $M5$ brane in the large N limit. The R-symmetry for this algebra is SO(5) corresponding to rotations in the 5 directions transverse to the brane. This is consistent with the formalism above because SO(5) $\sim$ Sp(4). We will use the symbols l_1, l_2 to represent the SO(5) Cartans in the orthogonal basis. The Sp(4) Cartans are given by:

$$k = \frac{l_1+l_2}{2}, \quad k_1 = \frac{l_1-l_2}{2} \tag{3.21}$$

Note, also that the bosonic part of the commuting subalgebra is $\mathrm{SU}(3,1) \otimes \mathrm{Sp}(2)$. In the calculation below, we will us the equivalence $\mathrm{Sp}(2) \sim \mathrm{SU}(2)$. The SU(2) charge is the same as the Sp(2) charge.

In the strict large N limit, the spectrum of this theory is the Fock space of supergravitons of M theory on $AdS_7 \times S^4$ [17, 18].[22] The set of primaries for the graviton spectrum is ($\epsilon_0 = 2p, l_1 = 2p, l_2 = 0, h_1 = 0, h_2 = 0, h_3 = 0$) [30],[23] where p can be any positive integer. Now given a highest weight state, we again use the procedure described in appendix A to obtain the representations (of the maximal compact subgroup) occurring in the supermultiplet. The result is enumerated in table 5 and agrees with [30]. By the action of momentum operators on this states we can build up the entire representation of the superconformal algebra.

It is now again simple to compute the Index over single gravitons once we have the spectrum. The Witten Index for the p^{th} graviton representation (R_p)(i.e. for a particular value of p in the primary), is obtained by

$$\begin{aligned} I^W_{R_p} =& Tr_{\Delta=0}\left[(-1)^F x^E z^{K_1} y_1^{H_1} y_2^{H_2}\right] \\ =& \sum_q \frac{(-1)^{2(h_2)_q} x^{(3\epsilon_0+h_1+h_2-h_3)_q} \chi_q^{\mathrm{SU}(2)}(z) \chi_q^{\mathrm{SU}(3)}(y_1,y_2)}{(1-x^4 y_1)(1-\frac{x^4 y_2}{y_1})(1-\frac{x^4}{y_2})}, \end{aligned} \tag{3.22}$$

where q runs over all the conformal representations with $\Delta = 0$ that appears in the decomposition of R_p in table 5; x, z, y_1 and y_2 are the exponential of the chemical potentials corresponding to the subgroup charges E, K_1, H_1 and H_2 respectively as defined in (3.14); $\chi^{\mathrm{SU}(2)}$ and $\chi^{\mathrm{SU}(3)}$ denote the characters of the groups SU(2) and SU(3) respectively, which are computed using the Weyl character formula.

The Index over the single particle states is then simply given by the following sum,

$$I^W_{sp} = \sum_{p=3}^{\infty} I^W_{R_p} + I^W_{R_2} + I^W_{R_1}, \tag{3.23}$$

Performed this sum, we find that the single particle contribution to the index is

$$\begin{aligned} I^W_{sp} &= \frac{\text{term1} + \text{term2}}{\text{den}} \\ \text{term1} &= x^6\left(\sqrt{z} y_1^2 \left(1 - x^8 y_2\right) x^2 + \sqrt{z} y_2 \left(1 - x^8 y_2\right) x^2\right) \\ \text{term2} &= x^6\left(y_1\left(-\sqrt{z} x^{10} + \sqrt{z} y_2^2 x^2 + \left(x^{12}-1\right)(z+1) y_2\right)\right) \\ \text{den} &= \left(\sqrt{z} x^{12} - (z+1) x^6 + \sqrt{z}\right)\left(x^4 y_1 - 1\right)\left(x^4 - y_2\right)\left(x^4 y_2 - y_1\right). \end{aligned} \tag{3.24}$$

The index over the Fock-space of gravitons can be obtained from the above single particle index by the formula (2.18).

[22] The index we will calculate is sensitive to $\frac{1}{16}$ BPS states. However, the $\frac{1}{4}$ BPS partition function has been calculated, even at finite N, in [19].

[23] we specify the highest weight of the maximal compact subgroup; ϵ_0 being the SO(2) charge, l_1 and l_2 being the SO(5) charges in orthogonal basis and h_1, h_2 and h_3 being the SO(6) charge also in the orthogonal basis.

[24] The '+' appears because the conformal representation we subtract is, itself short. See [22] for details

range of p	ϵ_0[SO(2)]	SO(6)[orth.]	SO(5)[orth.]	Δ	contribution
$p \geq 1$	$2p$	$(0,0,0)$	$(p,0)$	0	$+$
$p \geq 1$	$2p+\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\frac{2p-1}{2},\frac{1}{2})$	0	$+$
$p \geq 1$	$2p+1$	$(1,1,1)$	$(p-1,0)$	2	$+$
$p \geq 2$	$2p+1$	$(1,0,0)$	$(p-1,1)$	0	$+$
$p \geq 2$	$2p+\frac{3}{2}$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2})$	$(\frac{(2p-3)}{2},\frac{1}{2})$	2	$+$
$p \geq 2$	$2p+2$	$(2,0,0)$	$(p-2,0)$	4	$+$
$p \geq 3$	$2p+\frac{3}{2}$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(\frac{2p-3}{2},\frac{3}{2})$	0	$+$
$p \geq 3$	$2p+2$	$(1,1,0)$	$(p-2,1)$	2	$+$
$p \geq 3$	$2p+\frac{5}{2}$	$(\frac{3}{2},\frac{1}{2},-\frac{1}{2})$	$(\frac{(2p-5)}{2},\frac{1}{2})$	4	$+$
$p \geq 3$	$2p+3$	$(1,1,-1)$	$(p-3,0)$	6	$+$
$p \geq 4$	$2p+2$	$(0,0,0)$	$(p-2,2)$	2	$+$
$p \geq 4$	$2p+\frac{5}{2}$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\frac{2p-5}{2},\frac{3}{2})$	4	$+$
$p \geq 4$	$2p+3$	$(1,0,0)$	$(p-3,1)$	6	$+$
$n \geq 4$	$2p+\frac{7}{2}$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(\frac{2p-7}{2},\frac{1}{2})$	8	$+$
$p \geq 4$	$2p+4$	$(0,0,0)$	$(p-4,0)$	12	$+$
$p=1$	$\frac{7}{2}$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	0	$-$
$p=1$	4	$(1,1,0)$	$(0,0)$	2	$-$
$p=1$	4	$(0,0,0)$	$(1,0)$	2	$-$
$p=1$	5	$(1,0,0)$	$(0,0)$	4	$+$[24]
$p=1$	6	$(0,0,0)$	$(0,0)$	6	$-$
$p=2$	6	$(0,0,0)$	$(1,1)$	2	$-$
$p=2$	$\frac{13}{2}$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	4	$-$
$p=2$	7	$(1,0,0)$	$(0,0)$	6	$-$

Table 5: d=6 graviton spectrum.

To get a sense for the formula, let us set $z, y_i \to 1$ in (3.24) leaving only $x \equiv e^{-\beta}$. We remind the reader that β is the chemical chemical potential corresponding to $E = 3\epsilon_0 + h_1 + h_2 - h_3$. This leads to

$$I^W_{sp}(x)\big|_{z,y_i\to 1} = \frac{x^6\left(2x^4+x^2+2\right)}{\left(x^8+x^6-x^2-1\right)^2}. \tag{3.25}$$

We note that in the high energy limit when $x \to 1$, I^W_{sp} in (3.25) becomes $I^W_{sp} = \frac{5}{144\beta^2}$. Then by the use of (2.18) we have,

$$I^W_{\rm fock} = \exp\frac{5\zeta(3)}{144\beta^2}. \tag{3.26}$$

Then the average value of $E = 3\epsilon_0 + h_1 + h_2 - h_3$ is given by,

$$E = -\frac{\partial \ln I^W_{\rm fock}}{\partial\beta} = \frac{5\zeta(3)}{72\beta^3}. \tag{3.27}$$

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If we define an entropy like quantity S by

$$I_{\rm fock}^{W} = \int dy \exp\left(-\beta y\right) \exp S^{\rm ind}(y), \tag{3.28}$$

we find,

$$S^{\rm ind}(E) = \frac{5\zeta(3)/48}{(5\zeta(3)/72)^{\frac{2}{3}}} E^{\frac{2}{3}}. \tag{3.29}$$

We can also do a similar analysis with the partition function instead of the index. The single particle partition function evaluated on the $\Delta = 0$ states with all the other chemical potentials except β set to zero is given by,

$$Z_{sp}(x) = {\rm tr}_{\Delta=0} x^{E} = \frac{-x^6\left(-2x^8+x^6+x^2-2\right)}{\left(1-x^2\right)^5\left(x^2+1\right)\left(x^4+x^2+1\right)^2} \tag{3.30}$$

The separate contribution of the bosonic and fermionic states to the partition function in (3.30) are as follows,

$$Z_{\rm sp}^{\rm bose}(x) = {\rm tr}_{\Delta=0 \text{ bosons}} = \frac{x^6\left(3x^{10}-x^6+2\right)}{\left(1-x^4\right)^3\left(1-x^6\right)^2} \tag{3.31}$$

$$Z_{\rm sp}^{\rm fermi}(x) = {\rm tr}_{\Delta=0 \text{ fermions}} = \frac{x^8\left(2x^{10}-x^4+3\right)}{\left(1-x^4\right)^3\left(1-x^6\right)^2} \tag{3.32}$$

An analysis similar to that done for the Index, yields for the above partition function

$$\ln Z_{\rm fock} = \sum_n \frac{Z_{\rm sp}^{\rm bose}(x^n) + (-1)^{n+1} Z_{\rm sp}^{\rm fermi}}{n} = \frac{7\zeta(6)}{2048\beta^5} \tag{3.33}$$

$$S(E) = \frac{21\zeta(6)/1024}{(35\zeta(6)/2048)^{\frac{5}{6}}} E^{\frac{5}{6}}, \tag{3.34}$$

which is again similar to that of a six dimensional gas for reasons that are similar to those explained below equation (2.29). Note, that in this case, we have 2 transverse supersymmetric scalars and 3 derivatives.

3.8 The Index on the worldvolume theory of a single $M5$ brane

We will now compute our index over the worldvolume theory of a single $M5$ brane. For this free theory, the single particle state content is just the representation corresponding to $p = 1$ in table 5 of the previous subsection. This means that it corresponds to the representation of the $d = 6$ superconformal group with the primary having charges $\epsilon_0 = 2$, SO(6) highest weights $[0, 0, 0]$ and R-symmetry SO(5) highest weight $[1, 0]$. Physically, this multiplet corresponds to the 5 transverse scalars, real fermions transforming as chiral spinors of both SO(6) and SO(5) and a self-dual two form $B_{\mu\nu}$. See [31, 23, 24] and references therein for more details. Using table 5, we calculate the Index over these states

$$\begin{aligned} I_{M_5}^{\rm sp}(x, z, y_1, y_2) &= {\rm Tr}\left[(-1)^F x^E z^{K_1} y_1^{H_1} y_2^{H_2}\right] \\ &= \frac{x^6(z^{\frac{1}{2}} + \frac{1}{z^{\frac{1}{2}}}) - x^8\left(y_2 + \frac{y_1}{y_2} + \frac{1}{y_1}\right) + x^{12}}{(1-x^4 y_1)\left(1 - x^4\frac{y_2}{y_1}\right)\left(1 - \frac{x^4}{y_2}\right)} \end{aligned} \tag{3.35}$$

Specializing to the chemical potentials $y_i \to 1, z \to 1$, the index simplifies to

$$I^{\rm sp}_{M_5}(x, z = 1, y_i = 1) = \frac{2x^6 - 3x^8 + x^{12}}{(1 - x^4)^3} \tag{3.36}$$

Multiparticling this index, to get the index over the Fock space on the M_2 brane, we find that

$$\begin{aligned} I_{M_5}(x, z = 1, y_i = 1) &= \exp \sum_n \frac{I^{\rm sp}_{M_5}(x^n, z = 1, y_i = 1)}{n} \\ &= \prod_{n_1,n_2,n_3} \frac{\left(1 - x^{8+4(n_1+n_2+n_3)}\right)^3}{\left(1 - x^{6+4(n_1+n_2+n_3)}\right)^2 \left(1 - x^{12+4(n_1+n_2+n_3)}\right)} \end{aligned} \tag{3.37}$$

At high temperatures $x \equiv e^{-\beta} \to 1$, we find

$$I_{M_5}|_{x\to 1, y_i=1} = \exp\left\{\frac{\pi^2}{32\beta}\right\} \tag{3.38}$$

The supersymmetric single particle partition function, on the other hand is given by

$$\begin{aligned} Z^{\rm sp,susy}_{M_5}(x, z, y_1, y_2) &= {\rm Tr}_{\Delta=0}\left[x^E z^{K_1} y_1^{H_1} y_2^{H_2}\right] \\ &= \frac{x^6(z^{\frac{1}{2}} + \frac{1}{z^{\frac{1}{2}}}) + x^8\left(y_2 + \frac{y_1}{y_2} + \frac{1}{y_1}\right) + x^{12}}{(1 - x^4 y_1)\left(1 - x^4\frac{y_2}{y_1}\right)\left(1 - \frac{x^4}{y_2}\right)} \end{aligned} \tag{3.39}$$

In particular, setting $z, y_i = 1$, we find

$$Z^{\rm sp,susy}_{M_5}(x, z = 1, y_i = 1) = \frac{2x^6 + 3x^8 + x^{12}}{(1 - x^4)^3} \tag{3.40}$$

with contributions from the bosons and fermions being

$$\begin{aligned} Z^{\rm sp,susy,bose}_{M_5}(x) &= {\rm tr}_{\Delta=0\ \rm bosons} x^E = \frac{2x^6 + x^{12}}{(1 - x^4)^3} \\ Z^{\rm sp,susy,fermi}_{M_5}(x) &= {\rm tr}_{\Delta=0\ \rm fermions} x^E = \frac{3x^8}{(1 - x^4)^3} \end{aligned} \tag{3.41}$$

Multiparticling this result, we find

$$\begin{aligned} Z_{M_5}(x, z = 1, y_i = 1) &= \exp \sum_n \frac{Z^{\rm sp,susy}_{M_5}(x^n, z = 1, y_i = 1)}{n} \\ &= \prod_{n_1,n_2,n_3} \frac{\left(1 + x^{8+4(n_1+n_2+n_3)}\right)^3}{\left(1 - x^{6+4(n_1+n_2+n_3)}\right)^2 \left(1 - x^{12+4(n_1+n_2+n_3)}\right)} \end{aligned} \tag{3.42}$$

At high temperatures $x \to 1$, we find that

$$Z_{M_5}(x \to 1, z = 1, y_i = 1) \approx \exp\left\{\frac{45\zeta(4)}{512\beta^3}\right\} \tag{3.43}$$

4. d=5

4.1 The superconformal algebra and its unitary representations

In $d = 5$, the bosonic part of the superconformal algebra is $\mathrm{SO}(5,2) \otimes \mathrm{SU}(2)$. Under the $\mathrm{SO}(5) \otimes \mathrm{SO}(2)$ subgroup of the conformal group the supersymmetry generators Q^i_μ $i = 1, \cdots, 4$ and $\mu = \pm\frac{1}{2}$ transform as the spinors of SO(5), with charge $\frac{1}{2}$ under SO(2). The R-symmetry group is SU(2) and μ above is an SU(2) index. We use k to represent the SU(2) Cartan. The SO(5) Cartans in the orthogonal basis are denoted by h_1, h_2. We will use ϵ_0 to represent the energy which is measured by the charge under SO(2). To lighten the notation, we will use the same symbols to represent the eigenvalues of states under these Cartans.

With these conventions the Qs have $\epsilon_0 = \frac{1}{2}, k = \pm\frac{1}{2}$ and SO(5) charges:

$$\begin{aligned} Q^1 &\to \left(\frac{1}{2}, \frac{1}{2}\right), \quad Q^2 \to \left(\frac{1}{2}, -\frac{1}{2}\right) \\ Q^3 &\to \left(-\frac{1}{2}, \frac{1}{2}\right), \quad Q^4 \to \left(-\frac{1}{2}, -\frac{1}{2}\right) \end{aligned} \tag{4.1}$$

The superconformal generators S^μ_i are the conjugates of Q^i_μ and therefore their charges are the negative of the charges above.

The anticommutator between Q and S is given by

$$\{S^\mu_i, Q^j_\nu\} \sim \delta^\mu_\nu \left(T^j_i\right) - \delta^j_i M^\mu_\nu \tag{4.2}$$

Here T^j_i and M^μ_ν are the SO(5,2) and SU(2) generators respectively.

As in the previous sections, by diagonalizing this operator one can determine when a descendant of the primary will have zero norm. Performing this analysis [4], one finds that short representations can exist when the highest weights of the primary satisfy one of the following conditions

$$\begin{aligned} \epsilon_0 =& h_1 + h_2 + 3k + 4 \quad \text{when } \mathrm{h}_1 \geq \mathrm{h}_2 \geq 0 \text{ and } \mathrm{k} \geq 0, \\ \epsilon_0 =& h_1 + 3k + 3, \quad \text{when } \mathrm{h}_2 = 0 \text{ and } \mathrm{k} \geq 0, \\ \epsilon_0 =& 3k, \quad \text{when } \mathrm{h}_1 = \mathrm{h}_2 = 0, \text{ and } \mathrm{k} \geq 0. \end{aligned} \tag{4.3}$$

The last two conditions give isolated short representations.

4.2 Null vectors and character decomposition of a long representation at the unitarity threshold

As in the case of $d = 3, 6$, and as explained in the previous section the short representations of $d = 5$ are also either *regular* or *isolated*. The energy of a *regular* short representation is given by $\epsilon_0 = h_1 + h_2 + 3k + 4$. Again the null states of such a representation transform in an irreducible representation of the algebra; for $h_1 \neq 0 \neq h_2$ the highest weight of the primary at the head of this null irreducible representation is given in terms of the highest weight of the primary of the representation itself by $\epsilon'_0 = \epsilon_0 + \frac{1}{2}, \quad k' = k + \frac{1}{2}, \quad h'_1 =$

$h_1 - \frac{1}{2}, \quad h_2' = h_2 - \frac{1}{2}$. We note that $\epsilon_0' - h_1' - h_2' - 3k' - 4 = \epsilon_0 - h_1 - h_2 - 3k - 4 = 0$, which shows that the null states also transform in a *regular* short representation. Thus a long representation at the edge of this unitarity bound has the same state content as the union of ordinary and null states of such a *regular* short representation. So we conclude that,

$$\begin{aligned}\lim_{\delta\to 0}\chi(h_1{+}h_2{+}3k{+}4{+}\delta, h_1, h_2, k] =&\chi(h_1+h_2+3k+4, h_1, h_2, k)\\ &+\chi\left(h_1+h_2+3k+\frac{9}{2}, h_1-\frac{1}{2}, h_2-\frac{1}{2}, k+\frac{1}{2}\right),\\ &\left(\text{with } \mathrm{h}_1 \geq \mathrm{h}_2 \geq \frac{1}{2} \text{ and } \mathrm{k} \geq 0\right).\end{aligned} \tag{4.4}$$

where $\chi(\epsilon_0, h_1, h_2, k)$ is the character of the irreducible representation with energy ϵ_0, SO(5) highest weights(in the orthogonal basis) (h_1, h_2) and SU(2) highest weight k.

Now when $h_1 \geq 1, h_2 = 0$ the null states of the regular short representation occur at level two and are characterized by a primary with the highest weights $\epsilon_0' = \epsilon_0 + 1, \quad k' = k+1, \quad h_1' = h_1 - 1, \quad h_2' = 0$. Now we note that $h_1' \neq 0, h_2' = 0$ and $\epsilon_0' - h_1' - 3k' - 3 = \epsilon_0 - h_1 - 3k - 4 = 0$, and so we conclude that the null states of such a type of *regular* short representation transform in an *isolated* short representation. Thus for a long representation at the edge of such a unitarity bound we have,

$$\begin{aligned}\lim_{\delta\to 0}\chi(h_1+3k+3+\delta, h_1, h_2=0, k) =&\chi(h_1+3k+3, h_1, h_2=0, k)\\ &+\chi(h_1+3k+4, h_1-1, h_2=0, k+1).\\ &h_1 \geq 1, k \geq 0\end{aligned} \tag{4.5}$$

Finally when $h_1 = 0 = h_2$ the null states of the *regular* short representation occur at level four and are labeled by a primary with the highest weight $\epsilon_0' = \epsilon_0{+}2, \quad k' = k{+}2, \quad h_1' = 0, \quad h_2' = 0$. Here we note that $h_1' = 0 = h_2'$ and $\epsilon_0' - 3k' = \epsilon_0 - 3k - 4 = 0$, which shows that the null states of this type of *regular* short representation again transforms in an *isolated* short representation but the *isolated* short representation encountered here is different from the one encountered in the previous paragraph. Thus for long representations at the edge of this unitarity bound we have,

$$\lim_{\delta\to 0}\chi(3k+\delta, h_1=0, h_2=0, k) = \chi(3k, 0, 0, k) + \chi(3k+2, 0, 0, k+2), \quad k \geq 0. \tag{4.6}$$

Thus we see that the *isolated* short representations (as defined in the previous subsection) are separated from other representations with the same SO(5) and SU(2) weights by a finite gap in energy so it is not possible to *approach* such representations with long representations and therefore we do not have any equivalent of (4.5) or (4.6) at energies near $h_1 = 3k + 3$ (when $h_1 \geq 1, h_2 = 0$) or near $3k$ (when $h_1 = 0 = h_2$) with $k \geq 0$ in both the cases.

For use below we define the following notation. Let $c(h_1, h_2, k)$ denote a regular short representation with SO(5) and SU(2) highest weights (h_1, h_2) and k respectively, and with

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$\epsilon_0 = h_1 + h_2 + 3k + 4$ (when $h_1 \geq h_2 \geq 0$). We now extend this notation to include isolated short representations.

- $c(h_1, -\frac{1}{2}, k)$ with $h_1 > 0$ and $k \geq -\frac{1}{2}$ denotes the representation with SO(5) weights $(h_1 - \frac{1}{2}, 0)$ and SU(2) quantum number $k + \frac{1}{2}$ and with $\epsilon_0 = h_1 + 3k + 4$.
- $c(-\frac{1}{2}, -\frac{1}{2}, k)$ with $k \geq -\frac{3}{2}$ denotes the representation with SO(5) weights $(0, 0)$ and SU(2) quantum number $k + \frac{3}{2}$ and $\epsilon_0 = 3k + \frac{9}{2}$.

4.3 Indices

As in the previous cases of $d = 3, 6$ for $d = 5$ an Index is defined to be any linear combination of multiplicities of short representations that evaluates to zero on every collection of collection of representations that appears on the r.h.s. of (4.4), (4.5) and (4.6).We now list these Indices.

1. The multiplicities of short representations which never appear on the R.H.S of (4.4), (4.5) and (4.6). These are $c(-\frac{1}{2}, -\frac{1}{2}, k)$ for $k = 0, -\frac{1}{2}, -1, -\frac{3}{2}$ and $c(h_1, -\frac{1}{2}, k)$ for all $h_1 > 0$ and $k = 0, -\frac{1}{2}$.
2. The complete list of indices constructed from linear combinations of the multiplicities of representations that appear on the r.h.s. of (4.4), (4.5) and (4.6) is given by,

$$I^{(1)}_{M_1,M_2} = \sum_{p=-1}^{2M_2} (-1)^{p+1} n\left\{c\left(M_1 + \frac{p}{2}, \frac{p}{2}, M_2 - \frac{p}{2}\right)\right\}, \tag{4.7}$$

   where $n\{R\}$ denotes the multiplicities of representations of type R, and the Index label M_1 and M_2 are the values of $h_1 - h_2 = h_1 - \frac{p}{2}$ and $h_2 + k = \frac{p}{2} + k$ for every regular representation that appears in the sum above. Here M_1 can be a integer greater than or equal to zero and M_2 is an integer or half integer greater than or equal to zero.

4.4 Minimally BPS states: distinguished supercharge and commuting superalgebra

We consider the special Q with charges ($h_1 = -\frac{1}{2}, h_2 = -\frac{1}{2}, k = \frac{1}{2}, \epsilon_0 = \frac{1}{2}$). Let $S = Q^\dagger$ then we have,

$$\Delta \equiv \{S, Q\} = \epsilon_0 - (h_1 + h_2 + 3K) \tag{4.8}$$

We are now interested in a partition function over states annihilated by this special Q. Such states transform in an irreducible representation of the subalgebra of the superconformal algebra that commutes with $\{Q, S, \Delta\}$. This subalgebra turns out to be SU(2, 1). Note that unlike $d = 3, 6$ this subalgebra is a bosonic lie algebra, and not a super lie algebra. In the subalgebra, we will label states by their weights under the Cartan elements H^s_1, H^s_2, which are defined in terms of the Cartans of the full algebra by:

$$H^s_1 = h_1 - h_2, \quad H^s_2 = \epsilon_0 + \frac{h_1 + h_2}{2}. \tag{4.9}$$

Here, h_1, h_2 are the Cartans of the SO(5) algebra in the orthogonal basis and ϵ_0 represents the charge under SO(2).

4.5 A trace formula for the general index and its character decomposition

We define the Witten Index,

$$I^w = \mathrm{Tr}_R[(-1)^F \exp(-\zeta\Delta + \mu G)], \tag{4.10}$$

where the trace being evaluated over any Hilbert Space that hosts a reducible or irreducible representation of the $d = 5$ superconformal algebra. Here G is any element of the subalgebra that commutes with the set $\{S, Q, \Delta\}$ and $F = 2h_1$. It is always possible to express G as a linear combination of the subalgebra Cartans (as given by (4.9)) by a similarity transformation. Once again, the Witten Index is independent of ζ.

It is easy to check that the Witten Index I^W evaluated on any representation A (reducible or irreducible) is given by,

$$\begin{aligned} I^W(A) = &\sum_{M_1,M_2} I^{(1)}_{M_1,M_2}\chi_{\mathrm{sub}}\left(M_1, \frac{3}{2}M_1 + 3(M_2+2)\right) \\ &+ \sum_{h_1(\geq\frac{1}{2});k=-\frac{1}{2},0} n\left\{c\left(h_1, -\frac{1}{2}, k\right)\right\}\chi_{\mathrm{sub}}\left(h_1 + \frac{1}{2}, \frac{3}{2}h_1 + 3k + \frac{21}{4}\right) \\ &+ \sum_{k=-\frac{3}{2},-1,-\frac{1}{2},0} n\left\{c\left(-\frac{1}{2}, -\frac{1}{2}, k\right)\right\}\chi_{\mathrm{sub}}\left(0, 3k + \frac{9}{2}\right) \end{aligned} \tag{4.11}$$

with $\chi_{\mathrm{sub}}(H_1^s, H_2^s)$ is the character of a representation of the subgroup, with highest weights (H_1^s, H_2^s) in the conventions described above.

In order to obtain (4.11) we have used,

$$I^{wi}(c(h_1, h_2, k)) = (-1)^{2h_2+1}\chi_{\mathrm{sub}}\left(h_1 - h_2, \frac{3}{2}(h_1 + h_2) + 3k + 6\right) \tag{4.12}$$

$$I^{wi}\left(c\left(h_1, -\frac{1}{2}, k\right)\right) = \chi_{\mathrm{sub}}\left(h_1 + \frac{1}{2}, \frac{3}{2}h_1 + 3k + \frac{21}{4}\right) \tag{4.13}$$

$$I^{wi}\left(c\left(-\frac{1}{2}, -\frac{1}{2}, k\right)\right) = \chi_{\mathrm{sub}}\left(0, 3k + \frac{9}{2}\right) \tag{4.14}$$

Note that the states with $\Delta = 0$ in any short representation (which are the states that contribute to the Witten Index), may be organized into a single irreducible representation of the subalgebra that commutes with Q. The quantum numbers of this subalgebra representation may be determined in terms of the quantum numbers of the parent short representation. For a *regular* short representation the primary of the full representation has $\Delta = 4$ so the highest weight state of the representation of the subalgebra is reached by acting on it with the supercharges Q_1, Q_2, Q_3 with the charges ($h_1 = \frac{1}{2}, h_2 = \frac{1}{2}, k = \frac{1}{2}, \epsilon_0 = \frac{1}{2}$), ($h_1 = \frac{1}{2}, h_2 = -\frac{1}{2}, k = \frac{1}{2}, \epsilon_0 = \frac{1}{2}$), ($h_1 = -\frac{1}{2}, h_2 = \frac{1}{2}, k = \frac{1}{2}, \epsilon_0 = \frac{1}{2}$). These have $\Delta = -2, -1, -1$ respectively. Similarly an *isolated* short representation of type $c(h_1, -\frac{1}{2}, k)$ with $h_1 > 0$ and $k \geq -\frac{1}{2}$ has $\Delta = 3$ and is acted upon by Q_1 and Q_2 in order to reach the highest weight state of the representation of the subalgebra. Finally the *isolated* short representations of type $c(-\frac{1}{2}, -\frac{1}{2}, k)$ with $k \geq -\frac{3}{2}$ have $\Delta = 0$ and are themselves the highest weight states of the representation of the subalgebra.

We finally note that every Index constructed in subsection §§ 4.3 appears as the coefficient of a distinct subalgebra character in (4.11). Thus I^W may be used to reconstruct all superconformal Indices of the algebra which makes it the most general Index that is possible to construct from the algebra alone.

5. Discussion

In this paper we have presented formulae for the most general superconformal Index for superconformal algebras in 3, 5 and 6 dimensions. Our work generalizes the analogous construction of an index for four dimensional conformal field theories presented in [7].

We hope that our work will find eventual use in the study of the space of superconformal field theories in 3, 5 and 6 dimensions. It has recently become clear that the space of superconformal field theories in four dimensions is much richer than previously suspected [32]. The space of superconformal field theories in $d = 3, 5, 6$ may be equally intricate, although this question has been less studied. As our index is constant on any connected component in the space of superconformal field theories, it may play a useful role in the study of this space.

In this paper we have also demonstrated that the most general superconformal index, in all the dimensions that we have studied, is captured by a simple trace formula. This observation may turn out to be useful as traces may easily be reformulated as path integrals, which in turn can sometimes be evaluated, using either perturbative techniques or localization arguments.

The two dimensional index — the elliptic genus — has played an important role in the understanding of black hole entropy from string theory. However the four dimensional index defined in [7] does not seem to capture the entropy of black holes in any obvious way. It would be interesting to know what the analogous situation in in 3 and 6 dimensions. It would certainly be interesting, for instance, if the index for the theory on the world volume of the M2 of M5 brane underwent a large N transition as a function of chemical potentials, to a phase whose index entropy scales like $N^{\frac{3}{2}}$ and N^3 respectively. As we currently lack a computable framework for multiple M2 or M5 branes we do not know if this happens; however see [25] for recent interesting progress in this respect.

In this connection we also note that the index for the weakly coupled Chern Simons theories studied in this paper does undergo a large N phase transition as a function of temperature. It would be interesting to have a holographic dual description of these phase transitions.

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A. The Racah-Speiser algorithm

In this appendix, we describe the Racah-Speiser algorithm, that may be used to determine the state content of the supergraviton representations described in tables 1 and 5. This appendix is out of the main line of this paper, since this state content may also be found in [20, 30, 21]

First, we remind the reader how irreducible representations of Lie Algebras, and affine Lie Algebras may be constructed using Verma modules [33, 34]. A nice description that is particularly applicable to our situation is provided in [35].

One starts by decomposing the algebra ($\mathcal{G}$) as:

$$\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- \tag{A.1}$$

where $\mathcal{G}^+(\mathcal{G}^-)$ corresponds to the positive (negative) roots of $\mathcal{G}$ and $\mathcal{H}$ is the Cartan subalgebra.

To construct the Verma module $\mathcal{V}$ corresponding to a lowest weight $|\Omega\rangle$, one considers the linear space made up of the states $P(\mathcal{G}^+)|\Omega\rangle$ where P is any polynomial of the positive generators.

One may calculate the character of this module,

$$\chi_{\mathcal{V}}(\mu) = \mathrm{tr}_{\mathcal{V}} e^{\mu\cdot\mathcal{H}} \tag{A.2}$$

where μ is a vector in the dual space of $\mathcal{H}$. The Weyl group $\mathcal{W}$ of the algebra has a natural action on $\mathcal{H}$ and this induces a natural action on μ. Finally, to obtain the character of the irreducibe representation $R(\Omega)$, one symmetrizes $\chi_{\mathcal{V}}$ with respect to $\mathcal{W}$.

$$\chi_{R(\Omega)} = \sum_{w\in\mathcal{W}} \chi_{\mathcal{V}}(w(\mu)). \tag{A.3}$$

One may now read off the list of states in R using $\chi_{R(\Omega)}$.

Let us elucidate the method above by constructing the character of a representation of SU(2) of weight j. If $J_\pm$ denote the raising and lowering operators and J_3 be the Cartan, then the Verma module corresponding to a lowest weight state of weight $|-j\rangle$ is spanned by the states $(J_+)^l|-j\rangle$ with $l = 0, 1, 2, \ldots$. The character for this Verma module is given by,

$$(\chi_{\mathcal{V}})_j(x) = \mathrm{tr} x^{J_3} = \sum_{l=0}^{\infty} x^{-j+l} = \frac{x^{-j+1}}{1-x}, \tag{A.4}$$

The Weyl group of SU(2) is $\mathbb{Z}_2$ which has two elements. One is just the identity. The other takes $x \to x^{-1}$. So the character of the irreducible representation corresponding to the highest weight j is given by

$$\chi_j(x) = \frac{x^{j+1}}{x-1} + \frac{x^{-j-1}}{x^{-1}-1} = \frac{x^{j+\frac{1}{2}} - x^{-j-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}. \tag{A.5}$$

The corresponding theory for superalgebras is not as well known but was developed following the work of Kac [36]. Its application to the 4 dimensional superconformal algebra

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may be found in [35, 16]. Here, although we do not have a proof of this algorithm from first principles, we have followed the natural generalization of the procedure described in [16, 37, 38] for superconformal algebras in $d = 4$.

Starting with a lowest weight state one acts on this state with all the 'raising' operators of the algebra(which includes the supersymmetry generators). Then, one discards null states and all their descendants as explained in the sections above. This process results in a Verma module.

The character of this Verma module is particularly easy to construct. Although the exact structure of null vectors may be quite complicated as, for example, in section 3.2, the charges characterizing the null state (which is all that is important for the character) are always obtained by adding the charges of a particular supercharge (or combination of supercharges) to the charges of the primary. So, the character of the Verma module may be obtained by counting all possible actions of supercharges except for the specific combinations that lead to null states or their descendants.

One now symmetrizes this character over the Weyl group of the *maximal compact subgroup* to obtain the character of the irreducible representation corresponding to our highest weight.

B. Charges

In this appendix, explicitly list the charges of the supersymmetry generators in the world-volume theory of the $M2$ and $M5$ branes and also for the superconformal algebra in $d = 5$. For the $M2$ brane, we have 16 supersymmetry generators 'Q'. We use the notation $[\epsilon_0, j, h_1, h_2, h_3, h_4]$, where ϵ_0 is the energy, j the SO(3) charge and h_1, h_2, h_3, h_4 are the SO(8) charges in the orthogonal basis (with a choice of Cartans in which the Qs are in the vector). With this notation, the Qs have charges

$$\begin{aligned}
Q_1 &= \left[\frac{1}{2},\frac{1}{2},1,0,0,0\right] \; ; \quad Q_2 = \left[\frac{1}{2},\frac{1}{2},-1,0,0,0\right], \\
Q_3 &= \left[\frac{1}{2},\frac{1}{2},0,1,0,0\right] \; ; \quad Q_4 = \left[\frac{1}{2},\frac{1}{2},0,-1,0,0\right], \\
Q_5 &= \left[\frac{1}{2},\frac{1}{2},0,0,1,0\right] \; ; \quad Q_6 = \left[\frac{1}{2},\frac{1}{2},0,0,-1,0\right], \\
Q_7 &= \left[\frac{1}{2},\frac{1}{2},0,0,0,1\right] \; ; \quad Q_8 = \left[\frac{1}{2},\frac{1}{2},0,0,0,-1\right], \\
Q_9 &= \left[\frac{1}{2},-\frac{1}{2},1,0,0,0\right] \; ; \; Q_{10} = \left[\frac{1}{2},-\frac{1}{2},-1,0,0,0\right], \\
Q_{11} &= \left[\frac{1}{2},-\frac{1}{2},0,1,0,0\right] \; ; \; Q_{12} = \left[\frac{1}{2},-\frac{1}{2},0,-1,0,0\right], \\
Q_{13} &= \left[\frac{1}{2},-\frac{1}{2},0,0,1,0\right] \; ; \; Q_{14} = \left[\frac{1}{2},-\frac{1}{2},0,0,-1,0\right], \\
Q_{15} &= \left[\frac{1}{2},-\frac{1}{2},0,0,0,1\right] \; ; \; Q_{16} = \left[\frac{1}{2},-\frac{1}{2},0,0,0,-1\right].
\end{aligned} \tag{B.1}$$

For the $M5$ brane, we again have 16 supercharges. Here, we use the notation $[\epsilon_0, h_1, h_2, h_3, l_1, l_2]$ where ϵ_0 is the SO(2) charge, h_1, h_2, h_3 are the SO(6) charges in the orthogonal basis and l_1, l_2 are the SO(5) charges in the orthogonal basis. With his notation, the Qs have charges

$$
\begin{aligned}
Q_1 &= \left[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right], & Q_2 &= \left[\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right],\\
Q_3 &= \left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right], & Q_4 &= \left[\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right],\\
Q_5 &= \left[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right], & Q_6 &= \left[\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right],\\
Q_7 &= \left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right], & Q_8 &= \left[\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right],\\
Q_9 &= \left[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right], & Q_{10} &= \left[\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right],\\
Q_{11} &= \left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right], & Q_{12} &= \left[\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right],\\
Q_{13} &= \left[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right], & Q_{14} &= \left[\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right],\\
Q_{15} &= \left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right], & Q_{16} &= \left[\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right].
\end{aligned}
\tag{B.2}
$$

Finally, we also specify the $d = 5$ supercharges. We specify their charges in the notation$[\epsilon_0, h_1, h_2, k]$, where ϵ_0 is the energy, h_1, h_2 are the SO(5) charges in the orthogonal basis and k is the Sp(2) R-symmetry charge.

$$
\begin{aligned}
Q_1 &= \left[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right], & Q_2 &= \left[\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]\\
Q_3 &= \left[\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right], & Q_4 &= \left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]\\
Q_5 &= \left[\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right], & Q_6 &= \left[\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]\\
Q_7 &= \left[\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right], & Q_8 &= \left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]
\end{aligned}
\tag{B.3}
$$

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